# Computer algebra independent integration tests

4-Trig-functions/4.6-Cosecant/4.6.1.2-d-csc-^n-a+b-csc-^m

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# Contents

# Chapter 1

# Introduction

This report gives the result of running the computer algebra independent integration problems. The listing of the problems are maintained by and can be downloaded from `https://rulebasedintegration.org`

The number of integrals in this report is [ 59 ]. This is test number [ 129 ].

## 1.1 Listing of CAS systems tested

The following systems were tested at this time.

1. Mathematica 12.3 (64 bit) on windows 10.
2. Rubi 4.16.1 in Mathematica 12.1 on windows 10.
3. Maple 2021.1 (64 bit) on windows 10.
4. Maxima 5.44 on Linux. (via sagemath 9.3)
5. Fricas 1.3.7 on Linux (via sagemath 9.3)
6. Giac/Xcas 1.7 on Linux. (via sagemath 9.3)
7. Sympy 1.8 under Python 3.8.8 using Anaconda distribution on Ubuntu.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 under windows 10 (64 bit)

Maxima, Fricas and Giac/Xcas were called from inside SageMath. This was done using SageMath integrate command by changing the name of the algorithm to use the different CAS systems.

Sympy was called directly using Python.

## 1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed.

If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

| System | solved | Failed |
|---|---|---|
| Rubi | % 100.00 ( 59 ) | % 0.00 ( 0 ) |
| Mathematica | % 89.83 ( 53 ) | % 10.17 ( 6 ) |
| Maple | % 69.49 ( 41 ) | % 30.51 ( 18 ) |
| Maxima | % 42.37 ( 25 ) | % 57.63 ( 34 ) |
| Fricas | % 69.49 ( 41 ) | % 30.51 ( 18 ) |
| Sympy | % 5.08 ( 3 ) | % 94.92 ( 56 ) |
| Giac | % 64.41 ( 38 ) | % 35.59 ( 21 ) |
| Mupad | % 55.93 ( 33 ) | % 44.07 ( 26 ) |

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

| grade | description |
|---|---|
| A | Integral was solved and antiderivative is optimal in quality and leaf size. |
| B | Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size. |
| C | Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons<br>1. antiderivative contains a hypergeometric function and the optimal antiderivative does not.<br>2. antiderivative contains a special function and the optimal antiderivative does not.<br>3. antiderivative contains the imaginary unit and the optimal antiderivative does not. |
| F | Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised. |

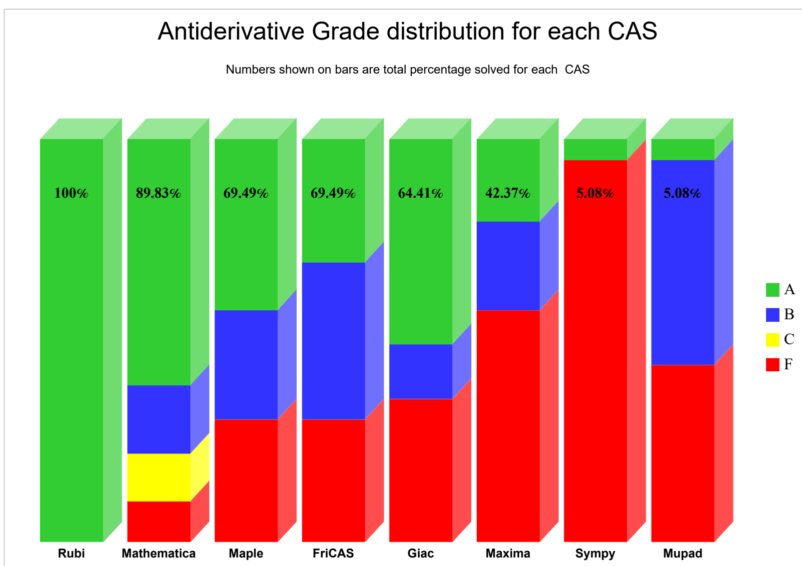
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

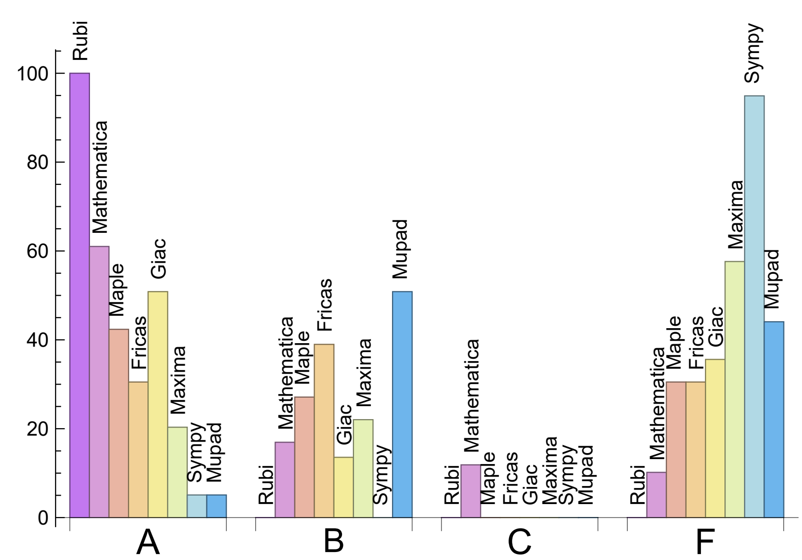
| System | % A grade | % B grade | % C grade | % F grade |
|---|---|---|---|---|
| Rubi | 100.00 | 0.00 | 0.00 | 0.00 |
| Mathematica | 61.02 | 16.95 | 11.86 | 10.17 |
| Maple | 42.37 | 27.12 | 0.00 | 30.51 |
| Maxima | 20.34 | 22.03 | 0.00 | 57.63 |
| Fricas | 30.51 | 38.98 | 0.00 | 30.51 |
| Sympy | 5.08 | 0.00 | 0.00 | 94.92 |
| Giac | 50.85 | 13.56 | 0.00 | 35.59 |
| Mupad | 5.08 | 50.85 | 0.00 | 44.07 |

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.




The figure below compares the CAS systems for each grade level.




The following table shows the distribution of the different types of failure for each CAS. There are 3 types of reasons why it can fail. The first is when CAS returns back the input within the time limit, which means it could not solve it. This the typical normal failure **F** .

The second is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned **F(-1)**.

The third is due to an exception generated. Assigned **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in CAS. This type of error requires more investigations to determine the cause.

| System | Number failed | Percentage normal failure | Percentage timeout failure | Percentage exception failure |
|---|---|---|---|---|
| Rubi | 0 | 0.00 % | 0.00 % | 0.00 % |
| Mathematica | 6 | 100.00 % | 0.00 % | 0.00 % |
| Maple | 18 | 100.00 % | 0.00 % | 0.00 % |
| Maxima | 34 | 61.76 % | 0.00 % | 38.24 % |
| Fricas | 18 | 100.00 % | 0.00 % | 0.00 % |
| Sympy | 56 | 94.64 % | 5.36 % | 0.00 % |
| Giac | 21 | 90.48 % | 0.00 % | 9.52 % |
| Mupad | 26 | 100.00 % | 0.00 % | 0.00 % |

Table 1.4: Time and leaf size performance for each CAS

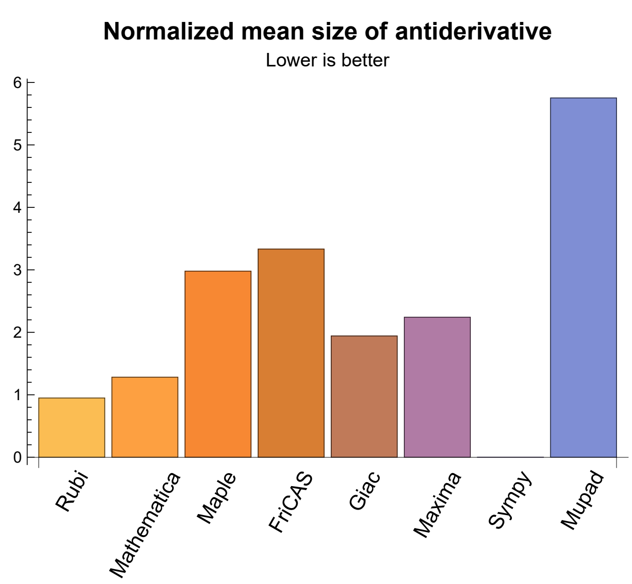
## 1.3 Performance

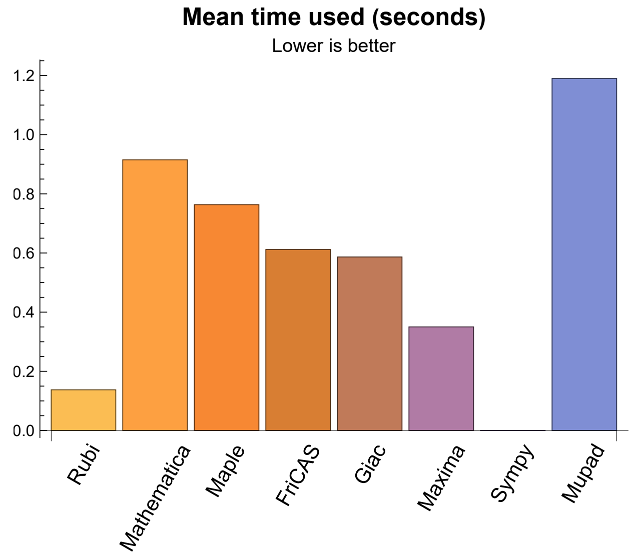
The table below summarizes the performance of each CAS system in terms of CPU time and leaf size of results.

| System | Mean time (sec) | Mean size | Normalized mean | Median size | Normalized median |
|---|---|---|---|---|---|
| Rubi | 0.14 | 113.10 | 0.95 | 69.00 | 1.00 |
| Mathematica | 0.92 | 95.26 | 1.28 | 76.00 | 1.16 |
| Maple | 0.76 | 248.12 | 2.98 | 100.00 | 1.53 |
| Maxima | 0.35 | 109.96 | 2.24 | 95.00 | 1.85 |
| Fricas | 0.61 | 252.10 | 3.33 | 181.00 | 3.05 |
| Sympy | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Giac | 0.59 | 127.71 | 1.94 | 81.50 | 1.52 |
| Mupad | 1.19 | 742.73 | 5.75 | 89.00 | 1.81 |

Table 1.5: Time and leaf size performance for each CAS

The following are bar charts for the normalized leafsize and time used columns from the above table.






## 1.4 list of integrals that has no closed form antiderivative

{57, 58, 59}

## 1.5 list of integrals solved by CAS but has no known antiderivative

**Rubi** {}

**Mathematica** {}

**Maple** {}

**Maxima** {}

**Fricas** {}

**Sympy** {}

**Giac** {}

**Mupad** {}

## 1.6 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not mean necessarily that the anti-derivative is wrong, as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it easier to do further investigation to determine why it was not possible to verify the result produced.

**Rubi** {}

**Mathematica** {}

**Maple** Verification phase not implemented yet.

**Maxima** Verification phase not implemented yet.

**Fricas** Verification phase not implemented yet.

**Sympy** Verification phase not implemented yet.

**Giac** Verification phase not implemented yet.

**Mupad** Verification phase not implemented yet.

## 1.7 Timing

The command `AboluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of _int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call has completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out is not counted in the final statistics.

## 1.8 Verification

A verification phase was applied on the result of integration for Rubi and Mathematica. Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative produced was correct.

Verification phase has 3 minutes time out. An integral whose result was not verified could still be correct. Further investigation is needed on those integrals which failed verifications. Such integrals are marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

## 1.9 Important notes about some of the results

### 1.9.1 Important note about Maxima results

Since these integrals are run in a batch mode, using an automated script, and by using sagemath (SageMath uses Maxima), then any integral where Maxima needs an interactive response from the user to answer a question during evaluation of the integral in order to complete the integration, will fail and is counted as failed.

The exception raised is `ValueError`. Therefore Maxima result below is lower than what could result if Maxima was run directly and each question Maxima asks was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the Timofeev test file, there were about 14 such integrals out of total 705, or about 2 percent. This pecrentage can be higher or lower depending on the specific input test file.

Such integrals can be indentified by looking at the output of the integration in each section for Maxima. The exception message will indicate of the error is due to the interactive question being asked or not.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
```

```
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath loading of Maxima `abs_integrate` was found to cause some problem. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/ for reference.

### 1.9.2 Important note about FriCAS and Giac/XCAS results

There are Few integrals which failed due to SageMath not able to translate the result back to SageMath syntax and not because these CAS system were not able to do the integrations.

These will fail With error `Exception raised: NotImplementedError`

The number of such cases seems to be very small. About 1 or 2 percent of all integrals.

Hopefully the next version of SageMath will have complete translation of FriCAS and XCAS syntax and I will re-run all the tests again when this happens.

### 1.9.3 Important note about finding leaf size of antiderivative

For Mathematica, Rubi and Maple, the buildin system function `LeafSize` is used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special buildin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy and Giac antiderivatives is determined using the following function, thanks to user `slelievre` at https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which is called directly from Python, the following code is used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
         leafCount =1
```

### 1.9.4 Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative, Maple was used to determine the leaf size of Mupad output by post processing.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

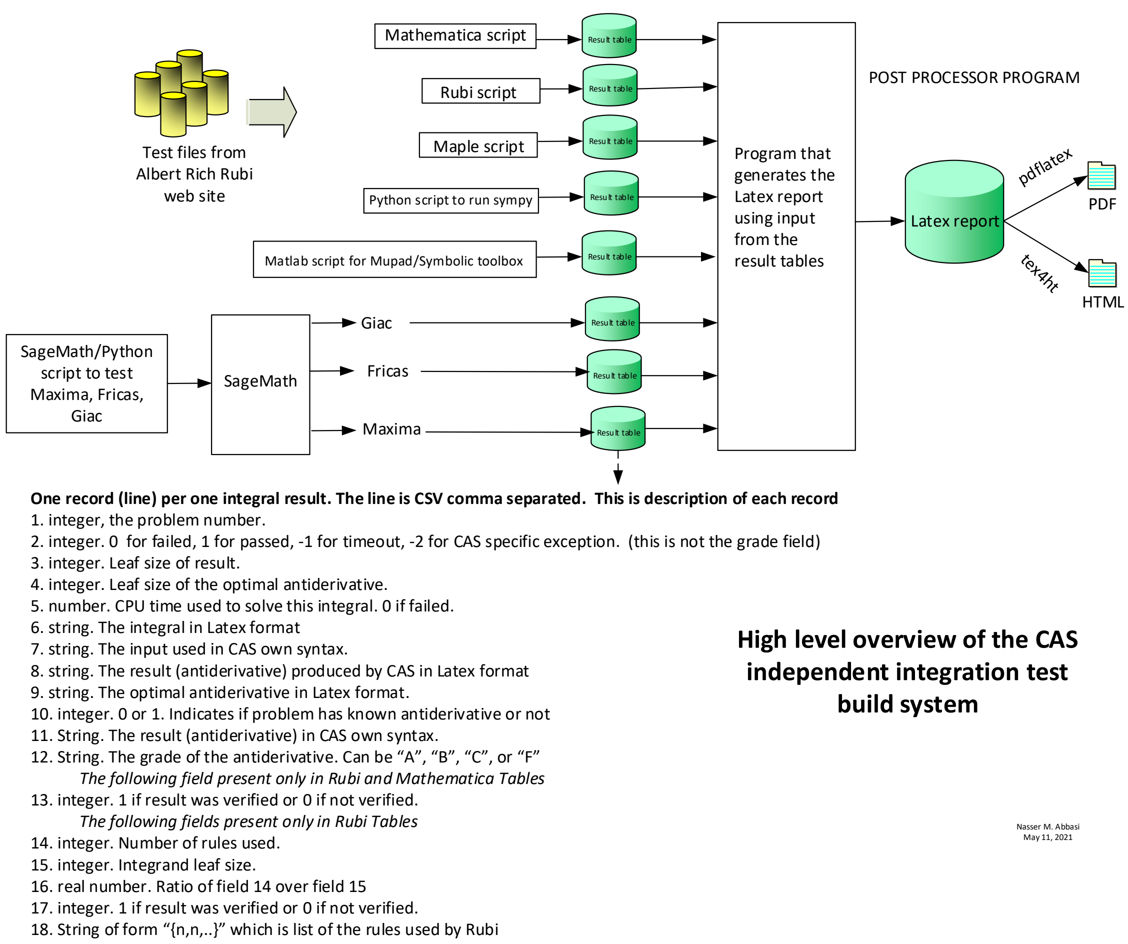
The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

## 1.10 Design of the test system

The following diagram gives a high level view of the current test build system.



# Chapter 2

# detailed summary tables of results

## 2.1 List of integrals sorted by grade for each CAS

### 2.1.1 Rubi

A grade: { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59 }

B grade: { }

C grade: { }

F grade: { }

### 2.1.2 Mathematica

A grade: { 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 28, 29, 30, 31, 32, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 53, 57, 58, 59 }

B grade: { 1, 3, 4, 5, 19, 20, 36, 37, 38, 52 }

C grade: { 21, 22, 23, 24, 25, 26, 27 }

F grade: { 33, 34, 35, 54, 55, 56 }

### 2.1.3 Maple

A grade: { 1, 2, 3, 4, 5, 6, 7, 11, 12, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 52, 53, 57, 58, 59 }

B grade: { 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 47, 48, 49, 50, 51 }

C grade: { }

F grade: { 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

### 2.1.4 Maxima

A grade: { 4, 5, 6, 16, 36, 37, 38, 52, 53, 57, 58, 59 }

B grade: { 1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17 }

C grade: { }

F grade: { 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 54, 55, 56 }

### 2.1.5 FriCAS

A grade: { 5, 6, 7, 8, 9, 10, 16, 43, 44, 45, 46, 47, 48, 52, 53, 57, 58, 59 }

B grade: { 1, 2, 3, 4, 11, 12, 13, 14, 15, 17, 18, 19, 20, 36, 37, 38, 39, 40, 41, 42, 49, 50, 51 }

C grade: { }

F grade: { 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

### 2.1.6 Sympy

A grade: { 57, 58, 59 }

B grade: { }

C grade: { }

F grade: { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56 }

### 2.1.7 Giac

A grade: { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 37, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 52, 53, 57, 58, 59 }

B grade: { 13, 14, 15, 17, 18, 36, 38, 51 }

C grade: { }

F grade: { 16, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

### 2.1.8 Mupad

A grade: { 57, 58, 59 }

B grade: { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53 }

C grade: { }

F grade: { 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

## 2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by table below. The elapsed time is in seconds. For failed result it is given as F(-1) if the failure was due to timeout. It is given as F(-2) if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given just an F.

In this table,the column **normalized size** is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$

| Problem 1 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | B | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 55 | 55 | 113 | 89 | 120 | 168 | 0 | 96 | 89 |
| normalized size | 1 | 1.00 | 2.05 | 1.62 | 2.18 | 3.05 | 0.00 | 1.75 | 1.62 |
| time (sec) | N/A | 0.071 | 0.817 | 0.280 | 0.323 | 0.541 | 0.000 | 0.334 | 0.350 |

| Problem 2 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | B | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 44 | 44 | 83 | 67 | 97 | 134 | 0 | 73 | 69 |
| normalized size | 1 | 1.00 | 1.89 | 1.52 | 2.20 | 3.05 | 0.00 | 1.66 | 1.57 |
| time (sec) | N/A | 0.066 | 0.345 | 0.274 | 0.330 | 0.475 | 0.000 | 0.365 | 0.256 |

| Problem 3 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | B | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 27 | 27 | 63 | 45 | 68 | 91 | 0 | 53 | 49 |
| normalized size | 1 | 1.00 | 2.33 | 1.67 | 2.52 | 3.37 | 0.00 | 1.96 | 1.81 |
| time (sec) | N/A | 0.087 | 0.159 | 0.228 | 0.330 | 0.518 | 0.000 | 0.560 | 0.253 |

| Problem 4 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | A | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 20 | 20 | 44 | 24 | 31 | 53 | 0 | 24 | 23 |
| normalized size | 1 | 1.00 | 2.20 | 1.20 | 1.55 | 2.65 | 0.00 | 1.20 | 1.15 |
| time (sec) | N/A | 0.056 | 0.054 | 0.248 | 0.323 | 0.489 | 0.000 | 0.597 | 0.183 |

| Problem 5 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | A | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 12 | 12 | 26 | 14 | 16 | 22 | 0 | 13 | 13 |
| normalized size | 1 | 1.00 | 2.17 | 1.17 | 1.33 | 1.83 | 0.00 | 1.08 | 1.08 |
| time (sec) | N/A | 0.020 | 0.024 | 0.204 | 0.336 | 0.501 | 0.000 | 0.445 | 0.172 |

| Problem 6 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 28 | 28 | 47 | 41 | 50 | 54 | 0 | 32 | 27 |
| normalized size | 1 | 1.00 | 1.68 | 1.46 | 1.79 | 1.93 | 0.00 | 1.14 | 0.96 |
| time (sec) | N/A | 0.013 | 0.093 | 0.525 | 0.416 | 0.556 | 0.000 | 0.422 | 0.216 |

| Problem 7 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | B | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 25 | 25 | 32 | 40 | 78 | 35 | 0 | 44 | 46 |
| normalized size | 1 | 1.00 | 1.28 | 1.60 | 3.12 | 1.40 | 0.00 | 1.76 | 1.84 |
| time (sec) | N/A | 0.047 | 0.077 | 0.396 | 0.426 | 0.610 | 0.000 | 0.534 | 0.236 |

| Problem 8 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 40 | 40 | 42 | 100 | 128 | 53 | 0 | 56 | 59 |
| normalized size | 1 | 1.00 | 1.05 | 2.50 | 3.20 | 1.32 | 0.00 | 1.40 | 1.48 |
| time (sec) | N/A | 0.061 | 0.131 | 0.522 | 0.429 | 0.519 | 0.000 | 0.685 | 0.273 |

| Problem 9 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 53 | 53 | 49 | 121 | 180 | 70 | 0 | 67 | 78 |
| normalized size | 1 | 1.00 | 0.92 | 2.28 | 3.40 | 1.32 | 0.00 | 1.26 | 1.47 |
| time (sec) | N/A | 0.067 | 0.172 | 0.507 | 0.428 | 0.449 | 0.000 | 0.289 | 0.347 |

| Problem 10 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 66 | 66 | 57 | 185 | 230 | 81 | 0 | 91 | 93 |
| normalized size | 1 | 1.00 | 0.86 | 2.80 | 3.48 | 1.23 | 0.00 | 1.38 | 1.41 |
| time (sec) | N/A | 0.073 | 0.251 | 0.434 | 0.425 | 0.587 | 0.000 | 0.344 | 0.402 |

| Problem 11 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | B | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 57 | 57 | 108 | 83 | 142 | 124 | 0 | 60 | 52 |
| normalized size | 1 | 1.00 | 1.89 | 1.46 | 2.49 | 2.18 | 0.00 | 1.05 | 0.91 |
| time (sec) | N/A | 0.066 | 0.325 | 0.709 | 0.427 | 0.474 | 0.000 | 0.736 | 0.406 |

| Problem 12 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | B | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 88 | 88 | 123 | 125 | 228 | 181 | 0 | 86 | 78 |
| normalized size | 1 | 1.00 | 1.40 | 1.42 | 2.59 | 2.06 | 0.00 | 0.98 | 0.89 |
| time (sec) | N/A | 0.110 | 1.038 | 0.794 | 0.436 | 0.520 | 0.000 | 0.449 | 2.178 |

| Problem 13 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 65 | 65 | 80 | 535 | 417 | 318 | 0 | 250 | -1 |
| normalized size | 1 | 1.00 | 1.23 | 8.23 | 6.42 | 4.89 | 0.00 | 3.85 | -0.02 |
| time (sec) | N/A | 0.091 | 1.902 | 0.920 | 0.474 | 0.565 | 0.000 | 1.010 | 0.000 |

| Problem 14 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 44 | 44 | 69 | 273 | 200 | 212 | 0 | 195 | -1 |
| normalized size | 1 | 1.00 | 1.57 | 6.20 | 4.55 | 4.82 | 0.00 | 4.43 | -0.02 |
| time (sec) | N/A | 0.030 | 0.089 | 0.631 | 0.446 | 0.456 | 0.000 | 1.162 | 0.000 |

| Problem 15 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 26 | 26 | 32 | 199 | 148 | 120 | 0 | 353 | -1 |
| normalized size | 1 | 1.00 | 1.23 | 7.65 | 5.69 | 4.62 | 0.00 | 13.58 | -0.04 |
| time (sec) | N/A | 0.016 | 0.054 | 0.716 | 0.442 | 0.546 | 0.000 | 1.055 | 0.000 |

| Problem 16 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | A | A | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 62 | 62 | 54 | 221 | 83 | 219 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.87 | 3.56 | 1.34 | 3.53 | 0.00 | 0.00 | -0.02 |
| time (sec) | N/A | 0.058 | 0.136 | 0.745 | 0.452 | 0.537 | 0.000 | 0.000 | 0.000 |

| Problem 17 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 81 | 81 | 129 | 1141 | 150 | 427 | 0 | 304 | -1 |
| normalized size | 1 | 1.00 | 1.59 | 14.09 | 1.85 | 5.27 | 0.00 | 3.75 | -0.01 |
| time (sec) | N/A | 0.102 | 0.428 | 0.811 | 0.452 | 0.605 | 0.000 | 1.930 | 0.000 |

| Problem 18 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 100 | 100 | 139 | 1961 | 0 | 546 | 0 | 348 | -1 |
| normalized size | 1 | 1.00 | 1.39 | 19.61 | 0.00 | 5.46 | 0.00 | 3.48 | -0.01 |
| time (sec) | N/A | 0.148 | 0.507 | 0.614 | 0.000 | 0.483 | 0.000 | 1.615 | 0.000 |

| Problem 19 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | B | F | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 37 | 37 | 108 | 113 | 0 | 283 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 2.92 | 3.05 | 0.00 | 7.65 | 0.00 | 0.00 | -0.03 |
| time (sec) | N/A | 0.059 | 0.400 | 2.354 | 0.000 | 0.804 | 0.000 | 0.000 | 0.000 |

| Problem 20 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | B | F | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 38 | 38 | 101 | 117 | 0 | 296 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 2.66 | 3.08 | 0.00 | 7.79 | 0.00 | 0.00 | -0.03 |
| time (sec) | N/A | 0.071 | 0.794 | 2.108 | 0.000 | 1.084 | 0.000 | 0.000 | 0.000 |

| Problem 21 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F(-1) | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 254 | 254 | 102 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.40 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.281 | 0.383 | 2.641 | 0.000 | 0.952 | 0.000 | 0.000 | 0.000 |

| Problem 22 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 213 | 213 | 46 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.22 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.128 | 0.230 | 4.174 | 0.000 | 0.906 | 0.000 | 0.000 | 0.000 |

| Problem 23 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 254 | 254 | 110 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.43 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.145 | 0.460 | 2.982 | 0.000 | 0.701 | 0.000 | 0.000 | 0.000 |

| Problem 24 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F(-1) | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 514 | 514 | 120 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.23 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.299 | 1.178 | 2.413 | 0.000 | 0.891 | 0.000 | 0.000 | 0.000 |

| Problem 25 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 470 | 470 | 109 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.23 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.253 | 0.968 | 2.178 | 0.000 | 1.454 | 0.000 | 0.000 | 0.000 |

| Problem 26 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 508 | 508 | 46 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.09 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.275 | 0.756 | 2.682 | 0.000 | 1.653 | 0.000 | 0.000 | 0.000 |

| Problem 27 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 552 | 552 | 72 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.307 | 1.218 | 1.804 | 0.000 | 1.888 | 0.000 | 0.000 | 0.000 |

| Problem 28 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 48 | 48 | 48 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.02 |
| time (sec) | N/A | 0.058 | 0.170 | 1.950 | 0.000 | 0.944 | 0.000 | 0.000 | 0.000 |

| Problem 29 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 69 | 69 | 73 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 1.06 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.070 | 0.990 | 2.448 | 0.000 | 0.646 | 0.000 | 0.000 | 0.000 |

| Problem 30 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 156 | 156 | 178 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 1.14 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.190 | 4.613 | 3.273 | 0.000 | 0.528 | 0.000 | 0.000 | 0.000 |

| Problem 31 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 109 | 109 | 126 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 1.16 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.103 | 1.348 | 1.997 | 0.000 | 0.582 | 0.000 | 0.000 | 0.000 |

| Problem 32 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 74 | 74 | 60 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.81 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.056 | 0.205 | 1.544 | 0.000 | 0.536 | 0.000 | 0.000 | 0.000 |

| Problem 33 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | F | F | F | F | F | F | F |
| verified | N/A | Yes | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 84 | 84 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.057 | 0.615 | 1.204 | 0.000 | 0.833 | 0.000 | 0.000 | 0.000 |

| Problem 34 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | F | F | F | F | F | F | F |
| verified | N/A | Yes | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 83 | 83 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.090 | 3.771 | 2.262 | 0.000 | 0.532 | 0.000 | 0.000 | 0.000 |

| Problem 35 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | F | F | F | F | F | F | F |
| verified | N/A | Yes | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 84 | 84 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.105 | 7.711 | 4.686 | 0.000 | 0.606 | 0.000 | 0.000 | 0.000 |

| Problem 36 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | A | B | F | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 107 | 107 | 568 | 139 | 125 | 217 | 0 | 205 | 314 |
| normalized size | 1 | 1.00 | 5.31 | 1.30 | 1.17 | 2.03 | 0.00 | 1.92 | 2.93 |
| time (sec) | N/A | 0.110 | 6.255 | 1.389 | 0.339 | 0.784 | 0.000 | 0.782 | 0.623 |

| Problem 37 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | A | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 73 | 73 | 152 | 99 | 95 | 155 | 0 | 134 | 234 |
| normalized size | 1 | 1.00 | 2.08 | 1.36 | 1.30 | 2.12 | 0.00 | 1.84 | 3.21 |
| time (sec) | N/A | 0.047 | 0.628 | 1.368 | 0.336 | 0.551 | 0.000 | 0.474 | 0.417 |

| Problem 38 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | A | B | F | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 34 | 34 | 76 | 52 | 43 | 77 | 0 | 74 | 105 |
| normalized size | 1 | 1.00 | 2.24 | 1.53 | 1.26 | 2.26 | 0.00 | 2.18 | 3.09 |
| time (sec) | N/A | 0.026 | 0.183 | 0.831 | 0.363 | 0.680 | 0.000 | 0.860 | 0.329 |

| Problem 39 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 112 | 112 | 125 | 162 | 0 | 607 | 0 | 194 | 588 |
| normalized size | 1 | 1.00 | 1.12 | 1.45 | 0.00 | 5.42 | 0.00 | 1.73 | 5.25 |
| time (sec) | N/A | 0.407 | 1.727 | 0.272 | 0.000 | 0.736 | 0.000 | 0.657 | 0.789 |

| Problem 40 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 84 | 84 | 144 | 112 | 0 | 524 | 0 | 141 | 515 |
| normalized size | 1 | 1.00 | 1.71 | 1.33 | 0.00 | 6.24 | 0.00 | 1.68 | 6.13 |
| time (sec) | N/A | 0.252 | 0.495 | 0.233 | 0.000 | 0.867 | 0.000 | 0.525 | 0.580 |

| Problem 41 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 62 | 62 | 106 | 77 | 0 | 308 | 0 | 98 | 135 |
| normalized size | 1 | 1.00 | 1.71 | 1.24 | 0.00 | 4.97 | 0.00 | 1.58 | 2.18 |
| time (sec) | N/A | 0.155 | 0.205 | 0.214 | 0.000 | 0.672 | 0.000 | 0.346 | 0.485 |

| Problem 42 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 53 | 53 | 62 | 53 | 0 | 245 | 0 | 63 | 129 |
| normalized size | 1 | 1.00 | 1.17 | 1.00 | 0.00 | 4.62 | 0.00 | 1.19 | 2.43 |
| time (sec) | N/A | 0.113 | 0.061 | 0.219 | 0.000 | 0.663 | 0.000 | 0.534 | 0.429 |

| Problem 43 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 40 | 40 | 40 | 39 | 0 | 154 | 0 | 48 | 36 |
| normalized size | 1 | 1.00 | 1.00 | 0.98 | 0.00 | 3.85 | 0.00 | 1.20 | 0.90 |
| time (sec) | N/A | 0.070 | 0.024 | 0.180 | 0.000 | 0.529 | 0.000 | 0.451 | 0.287 |

| Problem 44 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 57 | 57 | 59 | 70 | 0 | 238 | 0 | 77 | 184 |
| normalized size | 1 | 1.00 | 1.04 | 1.23 | 0.00 | 4.18 | 0.00 | 1.35 | 3.23 |
| time (sec) | N/A | 0.065 | 0.106 | 0.567 | 0.000 | 0.614 | 0.000 | 0.316 | 0.452 |

| Problem 45 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 61 | 61 | 56 | 72 | 0 | 235 | 0 | 77 | 766 |
| normalized size | 1 | 1.00 | 0.92 | 1.18 | 0.00 | 3.85 | 0.00 | 1.26 | 12.56 |
| time (sec) | N/A | 0.107 | 0.095 | 0.457 | 0.000 | 0.492 | 0.000 | 0.520 | 0.652 |

| Problem 46 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 82 | 82 | 78 | 142 | 0 | 285 | 0 | 112 | 1147 |
| normalized size | 1 | 1.00 | 0.95 | 1.73 | 0.00 | 3.48 | 0.00 | 1.37 | 13.99 |
| time (sec) | N/A | 0.261 | 0.113 | 0.488 | 0.000 | 0.676 | 0.000 | 0.637 | 0.840 |

| Problem 47 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | A | F(-1) | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 110 | 110 | 98 | 213 | 0 | 329 | 0 | 149 | 1218 |
| normalized size | 1 | 1.00 | 0.89 | 1.94 | 0.00 | 2.99 | 0.00 | 1.35 | 11.07 |
| time (sec) | N/A | 0.398 | 0.232 | 0.436 | 0.000 | 0.617 | 0.000 | 0.609 | 0.945 |

| Problem 48 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 144 | 144 | 129 | 405 | 0 | 410 | 0 | 252 | 1639 |
| normalized size | 1 | 1.00 | 0.90 | 2.81 | 0.00 | 2.85 | 0.00 | 1.75 | 11.38 |
| time (sec) | N/A | 0.588 | 0.308 | 0.378 | 0.000 | 1.131 | 0.000 | 0.662 | 1.258 |

| Problem 49 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 108 | 108 | 139 | 247 | 0 | 493 | 0 | 158 | 2677 |
| normalized size | 1 | 1.00 | 1.29 | 2.29 | 0.00 | 4.56 | 0.00 | 1.46 | 24.79 |
| time (sec) | N/A | 0.171 | 0.445 | 0.691 | 0.000 | 0.612 | 0.000 | 0.530 | 4.179 |

| Problem 50 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | B | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 170 | 170 | 216 | 796 | 0 | 933 | 0 | 297 | 5917 |
| normalized size | 1 | 1.00 | 1.27 | 4.68 | 0.00 | 5.49 | 0.00 | 1.75 | 34.81 |
| time (sec) | N/A | 0.319 | 1.079 | 0.775 | 0.000 | 0.754 | 0.000 | 0.381 | 8.803 |

| Problem 51 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | B | F | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 239 | 239 | 279 | 1912 | 0 | 1554 | 0 | 535 | 8167 |
| normalized size | 1 | 1.00 | 1.17 | 8.00 | 0.00 | 6.50 | 0.00 | 2.24 | 34.17 |
| time (sec) | N/A | 0.504 | 2.025 | 0.823 | 0.000 | 0.754 | 0.000 | 0.378 | 12.384 |

| Problem 52 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | A | A | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 31 | 31 | 66 | 36 | 49 | 33 | 0 | 49 | 39 |
| normalized size | 1 | 1.00 | 2.13 | 1.16 | 1.58 | 1.06 | 0.00 | 1.58 | 1.26 |
| time (sec) | N/A | 0.029 | 0.051 | 0.565 | 0.413 | 0.510 | 0.000 | 0.586 | 0.245 |

| Problem 53 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 68 | 68 | 67 | 53 | 71 | 52 | 0 | 45 | 27 |
| normalized size | 1 | 1.00 | 0.99 | 0.78 | 1.04 | 0.76 | 0.00 | 0.66 | 0.40 |
| time (sec) | N/A | 0.039 | 0.045 | 0.635 | 0.410 | 0.538 | 0.000 | 0.509 | 0.291 |

| Problem 54 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | F | F | F | F | F | F | F |
| verified | N/A | Yes | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 274 | 274 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.343 | 4.514 | 2.102 | 0.000 | 0.623 | 0.000 | 0.000 | 0.000 |

| Problem 55 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | F | F | F | F | F | F | F |
| verified | N/A | Yes | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 220 | 220 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.226 | 2.770 | 1.316 | 0.000 | 0.532 | 0.000 | 0.000 | 0.000 |

| Problem 56 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | F | F | F | F | F | F | F |
| verified | N/A | Yes | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 104 | 104 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.074 | 1.972 | 1.721 | 0.000 | 0.508 | 0.000 | 0.000 | 0.000 |

| Problem 57 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | N/A | A | A | A | A | A | A | A | A |
| verified | N/A | N/A | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.07 |
| time (sec) | N/A | 0.010 | 1.504 | 1.238 | 0.000 | 0.568 | 0.000 | 0.000 | 0.000 |

| Problem 58 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | N/A | A | A | A | A | A | A | A | A |
| verified | N/A | N/A | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.05 |
| time (sec) | N/A | 0.032 | 6.362 | 1.067 | 0.000 | 0.504 | 0.000 | 0.000 | 0.000 |

| Problem 59 | Optimal | Rubi | Mathematica | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | N/A | A | A | A | A | A | A | A | A |
| verified | N/A | N/A | N/A | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |
| normalized size | 1 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.04 |
| time (sec) | N/A | 0.040 | 6.238 | 4.447 | 0.000 | 0.505 | 0.000 | 0.000 | 0.000 |

## 2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is given. The larger this ratio is, the harder the integral was to solve. In this test, problem number [39] had the largest ratio of [.6923]

Table 2.1: Rubi specific breakdown of results for each integral

| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 1 | A | 6 | 5 | 1.00 | 13 | 0.385 |
| 2 | A | 6 | 6 | 1.00 | 13 | 0.462 |
| 3 | A | 4 | 4 | 1.00 | 13 | 0.308 |
| 4 | A | 3 | 3 | 1.00 | 13 | 0.231 |
| 5 | A | 1 | 1 | 1.00 | 11 | 0.091 |
| 6 | A | 2 | 2 | 1.00 | 12 | 0.167 |
| 7 | A | 4 | 4 | 1.00 | 11 | 0.364 |
| 8 | A | 5 | 5 | 1.00 | 13 | 0.385 |
| 9 | A | 6 | 5 | 1.00 | 13 | 0.385 |
| 10 | A | 7 | 5 | 1.00 | 13 | 0.385 |
| 11 | A | 3 | 3 | 1.00 | 12 | 0.250 |
| 12 | A | 4 | 4 | 1.00 | 12 | 0.333 |
| 13 | A | 5 | 5 | 1.00 | 10 | 0.500 |
| 14 | A | 4 | 4 | 1.00 | 10 | 0.400 |
| 15 | A | 2 | 2 | 1.00 | 10 | 0.200 |
| 16 | A | 5 | 4 | 1.00 | 10 | 0.400 |
| 17 | A | 6 | 5 | 1.00 | 10 | 0.500 |
| 18 | A | 7 | 6 | 1.00 | 10 | 0.600 |
| 19 | A | 2 | 2 | 1.00 | 25 | 0.080 |
| 20 | A | 2 | 2 | 1.00 | 28 | 0.071 |
| 21 | A | 4 | 4 | 1.00 | 25 | 0.160 |
| 22 | A | 3 | 3 | 1.00 | 25 | 0.120 |
| 23 | A | 4 | 4 | 1.00 | 25 | 0.160 |
| 24 | A | 6 | 6 | 1.00 | 25 | 0.240 |
| 25 | A | 5 | 5 | 1.00 | 25 | 0.200 |
| 26 | A | 6 | 6 | 1.00 | 25 | 0.240 |
| 27 | A | 7 | 6 | 1.00 | 25 | 0.240 |

Continued on next page

Table 2.1 – continued from previous page

| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 28 | A | 2 | 2 | 1.00 | 23 | 0.087 |
| 29 | A | 3 | 3 | 1.00 | 24 | 0.125 |
| 30 | A | 5 | 5 | 1.00 | 21 | 0.238 |
| 31 | A | 4 | 4 | 1.00 | 21 | 0.190 |
| 32 | A | 3 | 3 | 1.00 | 19 | 0.158 |
| 33 | A | 3 | 3 | 1.00 | 12 | 0.250 |
| 34 | A | 3 | 3 | 1.00 | 19 | 0.158 |
| 35 | A | 3 | 3 | 1.00 | 21 | 0.143 |
| 36 | A | 6 | 5 | 1.00 | 12 | 0.417 |
| 37 | A | 5 | 4 | 1.00 | 12 | 0.333 |
| 38 | A | 4 | 4 | 1.00 | 12 | 0.333 |
| 39 | A | 9 | 9 | 1.00 | 13 | 0.692 |
| 40 | A | 8 | 8 | 1.00 | 13 | 0.615 |
| 41 | A | 7 | 7 | 1.00 | 13 | 0.538 |
| 42 | A | 6 | 6 | 1.00 | 13 | 0.462 |
| 43 | A | 4 | 4 | 1.00 | 11 | 0.364 |
| 44 | A | 4 | 4 | 1.00 | 12 | 0.333 |
| 45 | A | 6 | 6 | 1.00 | 11 | 0.546 |
| 46 | A | 7 | 7 | 1.00 | 13 | 0.538 |
| 47 | A | 8 | 7 | 1.00 | 13 | 0.538 |
| 48 | A | 9 | 7 | 1.00 | 13 | 0.538 |
| 49 | A | 6 | 6 | 1.00 | 12 | 0.500 |
| 50 | A | 7 | 7 | 1.00 | 12 | 0.583 |
| 51 | A | 8 | 7 | 1.00 | 12 | 0.583 |
| 52 | A | 2 | 2 | 1.00 | 12 | 0.167 |
| 53 | A | 5 | 4 | 1.00 | 12 | 0.333 |
| 54 | A | 8 | 5 | 1.00 | 21 | 0.238 |
| 55 | A | 7 | 4 | 1.00 | 21 | 0.190 |
| 56 | A | 3 | 3 | 1.00 | 19 | 0.158 |
| 57 | A | 0 | 0 | 0.00 | 0 | 0.000 |
| 58 | A | 0 | 0 | 0.00 | 0 | 0.000 |
| 59 | A | 0 | 0 | 0.00 | 0 | 0.000 |

# Chapter 3

# Listing of integrals

## 3.1 $\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=55

$$-\frac{4\cot^3(x)}{3a}-\frac{4\cot(x)}{a}+\frac{3\tanh^{-1}(\cos(x))}{2a}+\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}+\frac{3\cot(x)\csc(x)}{2a}$$

```
[Out] 3/2*arctanh(cos(x))/a-4*cot(x)/a-4/3*cot(x)^3/a+3/2*cot(x)*csc(x)/a+cot(x)*
csc(x)^3/(a+a*csc(x))
```

**Rubi** [A] time = 0.07, antiderivative size = 55, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {3818, 3787, 3768, 3770, 3767}

$$-\frac{4\cot^3(x)}{3a}-\frac{4\cot(x)}{a}+\frac{3\tanh^{-1}(\cos(x))}{2a}+\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}+\frac{3\cot(x)\csc(x)}{2a}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^5/(a + a*Csc[x]),x]
```

```
[Out] (3*ArcTanh[Cos[x]])/(2*a) - (4*Cot[x])/a - (4*Cot[x]^3)/(3*a) + (3*Cot[x]*C
sc[x])/(2*a) + (Cot[x]*Csc[x]^3)/(a + a*Csc[x])
```

Rule 3767

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rule 3768

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :> -Simp[(b*Cos[c + d*x
]*(b*Csc[c + d*x])^(n - 1))/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), I
nt[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] &&
IntegerQ[2*n]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3787

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Dist[a, Int[(d*Csc[e + f*x])^n, x], x] + Dist[b/d, Int[
(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

Rule 3818

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(d^2*Cot[e + f*x]*(d*Csc[e + f*x])^(n - 2))/(f*(a +
 b*Csc[e + f*x])), x] - Dist[d^2/(a*b), Int[(d*Csc[e + f*x])^(n - 2)*(b*(n
- 2) - a*(n - 1)*Csc[e + f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[
a^2 - b^2, 0] && GtQ[n, 1]
```

Rubi steps

$$
\begin{aligned}
\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx &= \frac{\cot(x)\csc^3(x)}{a+a\csc(x)}-\frac{\int \csc^3(x)(3a-4a\csc(x))\,dx}{a^2}\\
&= \frac{\cot(x)\csc^3(x)}{a+a\csc(x)}-\frac{3\int \csc^3(x)\,dx}{a}+\frac{4\int \csc^4(x)\,dx}{a}\\
&= \frac{3\cot(x)\csc(x)}{2a}+\frac{\cot(x)\csc^3(x)}{a+a\csc(x)}-\frac{3\int \csc(x)\,dx}{2a}-\frac{4\,\mathrm{Subst}\left(\int \left(1+x^2\right)dx,x,\cot(x)\right)}{a}\\
&= \frac{3\tanh^{-1}(\cos(x))}{2a}-\frac{4\cot(x)}{a}-\frac{4\cot^3(x)}{3a}+\frac{3\cot(x)\csc(x)}{2a}+\frac{\cot(x)\csc^3(x)}{a+a\csc(x)}
\end{aligned}
$$

**Mathematica [B]** time = 0.82, size = 113, normalized size = 2.05

$$
\frac{20\tan\left(\frac{x}{2}\right)-20\cot\left(\frac{x}{2}\right)+3\csc^2\left(\frac{x}{2}\right)-3\sec^2\left(\frac{x}{2}\right)-36\log\left(\sin\left(\frac{x}{2}\right)\right)+36\log\left(\cos\left(\frac{x}{2}\right)\right)+\frac{48\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}-\frac{1}{2}\sin(x)\,\mathrm{cs}}{24a}
$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^5/(a + a*Csc[x]),x]
```

```
[Out] (-20*Cot[x/2] + 3*Csc[x/2]^2 + 36*Log[Cos[x/2]] - 36*Log[Sin[x/2]] - 3*Sec[
x/2]^2 + 8*Csc[x]^3*Sin[x/2]^4 + (48*Sin[x/2])/(Cos[x/2] + Sin[x/2]) - (Csc
[x/2]^4*Sin[x])/2 + 20*Tan[x/2])/(24*a)
```

**fricas [B]** time = 0.54, size = 168, normalized size = 3.05

$$
\frac{32\cos(x)^4+14\cos(x)^3-48\cos(x)^2+9\left(\cos(x)^4-2\cos(x)^2-\left(\cos(x)^3+\cos(x)^2-\cos(x)-1\right)\sin(x)+1\right)\log}{12\left(a\,\mathrm{c}\right.}
$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^5/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] 1/12*(32*cos(x)^4 + 14*cos(x)^3 - 48*cos(x)^2 + 9*(cos(x)^4 - 2*cos(x)^2 -
(cos(x)^3 + cos(x)^2 - cos(x) - 1)*sin(x) + 1)*log(1/2*cos(x) + 1/2) - 9*(c
os(x)^4 - 2*cos(x)^2 - (cos(x)^3 + cos(x)^2 - cos(x) - 1)*sin(x) + 1)*log(-
1/2*cos(x) + 1/2) + 2*(16*cos(x)^3 + 9*cos(x)^2 - 15*cos(x) - 6)*sin(x) - 1
8*cos(x) + 12)/(a*cos(x)^4 - 2*a*cos(x)^2 - (a*cos(x)^3 + a*cos(x)^2 - a*co
s(x) - a)*sin(x) + a)
```

**giac [A]** time = 0.33, size = 96, normalized size = 1.75

$$-\frac{3\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{2a}+\frac{a^2\tan\left(\frac{1}{2}x\right)^3-3a^2\tan\left(\frac{1}{2}x\right)^2+21a^2\tan\left(\frac{1}{2}x\right)}{24a^3}-\frac{2}{a\left(\tan\left(\frac{1}{2}x\right)+1\right)}+\frac{66\tan\left(\frac{1}{2}x\right)^3-21\,\mathrm{t}}{24\,a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^5/(a+a*csc(x)),x, algorithm="giac")
```

```
[Out] -3/2*log(abs(tan(1/2*x)))/a + 1/24*(a^2*tan(1/2*x)^3 - 3*a^2*tan(1/2*x)^2 +
 21*a^2*tan(1/2*x))/a^3 - 2/(a*(tan(1/2*x) + 1)) + 1/24*(66*tan(1/2*x)^3 -
21*tan(1/2*x)^2 + 3*tan(1/2*x) - 1)/(a*tan(1/2*x)^3)
```

**maple [A]** time = 0.28, size = 89, normalized size = 1.62

$$\frac{\tan^3\left(\frac{x}{2}\right)}{24a}-\frac{\tan^2\left(\frac{x}{2}\right)}{8a}+\frac{7\tan\left(\frac{x}{2}\right)}{8a}-\frac{1}{24a\tan\left(\frac{x}{2}\right)^3}+\frac{1}{8a\tan\left(\frac{x}{2}\right)^2}-\frac{7}{8a\tan\left(\frac{x}{2}\right)}-\frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^5/(a+a*csc(x)),x)
```

```
[Out] 1/24/a*tan(1/2*x)^3-1/8/a*tan(1/2*x)^2+7/8/a*tan(1/2*x)-1/24/a/tan(1/2*x)^3
+1/8/a/tan(1/2*x)^2-7/8/a/tan(1/2*x)-3/2/a*ln(tan(1/2*x))-2/a/(tan(1/2*x)+1
)
```

**maxima [B]** time = 0.32, size = 120, normalized size = 2.18

$$\frac{\frac{21\sin(x)}{\cos(x)+1}-\frac{3\sin(x)^2}{(\cos(x)+1)^2}+\frac{\sin(x)^3}{(\cos(x)+1)^3}}{24\,a}+\frac{\frac{2\sin(x)}{\cos(x)+1}-\frac{18\sin(x)^2}{(\cos(x)+1)^2}-\frac{69\sin(x)^3}{(\cos(x)+1)^3}-1}{24\left(\frac{a\sin(x)^3}{(\cos(x)+1)^3}+\frac{a\sin(x)^4}{(\cos(x)+1)^4}\right)}-\frac{3\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{2\,a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^5/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] 1/24*(21*sin(x)/(cos(x) + 1) - 3*sin(x)^2/(cos(x) + 1)^2 + sin(x)^3/(cos(x)
 + 1)^3)/a + 1/24*(2*sin(x)/(cos(x) + 1) - 18*sin(x)^2/(cos(x) + 1)^2 - 69*
sin(x)^3/(cos(x) + 1)^3 - 1)/(a*sin(x)^3/(cos(x) + 1)^3 + a*sin(x)^4/(cos(x
) + 1)^4) - 3/2*log(sin(x)/(cos(x) + 1))/a
```

**mupad [B]** time = 0.35, size = 89, normalized size = 1.62

$$\frac{7\tan\left(\frac{x}{2}\right)}{8\,a}-\frac{23\tan\left(\frac{x}{2}\right)^3+6\tan\left(\frac{x}{2}\right)^2-\frac{2\tan\left(\frac{x}{2}\right)}{3}+\frac{1}{3}}{8\,a\tan\left(\frac{x}{2}\right)^4+8\,a\tan\left(\frac{x}{2}\right)^3}-\frac{\tan\left(\frac{x}{2}\right)^2}{8\,a}+\frac{\tan\left(\frac{x}{2}\right)^3}{24\,a}-\frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2\,a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^5*(a + a/sin(x))),x)
```

```
[Out] (7*tan(x/2))/(8*a) - (6*tan(x/2)^2 - (2*tan(x/2))/3 + 23*tan(x/2)^3 + 1/3)/
(8*a*tan(x/2)^3 + 8*a*tan(x/2)^4) - tan(x/2)^2/(8*a) + tan(x/2)^3/(24*a) -
(3*log(tan(x/2)))/(2*a)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int\frac{\csc^5(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**5/(a+a*csc(x)),x)
```

```
[Out] Integral(csc(x)**5/(csc(x) + 1), x)/a
```

## 3.2 $\int \frac{\csc^4(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=44

$$\frac{2\cot(x)}{a}-\frac{3\tanh^{-1}(\cos(x))}{2a}+\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{3\cot(x)\csc(x)}{2a}$$

```
[Out] -3/2*arctanh(cos(x))/a+2*cot(x)/a-3/2*cot(x)*csc(x)/a+cot(x)*csc(x)^2/(a+a*
csc(x))
```

**Rubi [A]** time = 0.07, antiderivative size = 44, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.462, Rules used = {3818, 3787, 3767, 8, 3768, 3770}

$$\frac{2\cot(x)}{a}-\frac{3\tanh^{-1}(\cos(x))}{2a}+\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{3\cot(x)\csc(x)}{2a}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^4/(a + a*Csc[x]),x]
```

```
[Out] (-3*ArcTanh[Cos[x]])/(2*a) + (2*Cot[x])/a - (3*Cot[x]*Csc[x])/(2*a) + (Cot[
x]*Csc[x]^2)/(a + a*Csc[x])
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 3767

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rule 3768

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :> -Simp[(b*Cos[c + d*x
]*(b*Csc[c + d*x])^(n - 1))/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), I
nt[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] &&
IntegerQ[2*n]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3787

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Dist[a, Int[(d*Csc[e + f*x])^n, x], x] + Dist[b/d, Int[
(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

Rule 3818

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(d^2*Cot[e + f*x]*(d*Csc[e + f*x])^(n - 2))/(f*(a +
 b*Csc[e + f*x])), x] - Dist[d^2/(a*b), Int[(d*Csc[e + f*x])^(n - 2)*(b*(n
- 2) - a*(n - 1)*Csc[e + f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[
a^2 - b^2, 0] && GtQ[n, 1]
```

Rubi steps

$$\int \frac{\csc^4(x)}{a+a\csc(x)}\,dx = \frac{\cot(x)\csc^2(x)}{a+a\csc(x)} - \frac{\int \csc^2(x)(2a-3a\csc(x))\,dx}{a^2}$$
$$= \frac{\cot(x)\csc^2(x)}{a+a\csc(x)} - \frac{2\int \csc^2(x)\,dx}{a} + \frac{3\int \csc^3(x)\,dx}{a}$$
$$= -\frac{3\cot(x)\csc(x)}{2a} + \frac{\cot(x)\csc^2(x)}{a+a\csc(x)} + \frac{3\int \csc(x)\,dx}{2a} + \frac{2\,\text{Subst}(\int 1\,dx, x, \cot(x))}{a}$$
$$= -\frac{3\tanh^{-1}(\cos(x))}{2a} + \frac{2\cot(x)}{a} - \frac{3\cot(x)\csc(x)}{2a} + \frac{\cot(x)\csc^2(x)}{a+a\csc(x)}$$

**Mathematica** [A] time = 0.35, size = 83, normalized size = 1.89

$$\frac{-4\tan\left(\frac{x}{2}\right)+4\cot\left(\frac{x}{2}\right)-\csc^2\left(\frac{x}{2}\right)+\sec^2\left(\frac{x}{2}\right)+12\log\left(\sin\left(\frac{x}{2}\right)\right)-12\log\left(\cos\left(\frac{x}{2}\right)\right)-\frac{16\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}}{8a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^4/(a + a*Csc[x]),x]
```

```
[Out] (4*Cot[x/2] - Csc[x/2]^2 - 12*Log[Cos[x/2]] + 12*Log[Sin[x/2]] + Sec[x/2]^2
 - (16*Sin[x/2])/(Cos[x/2] + Sin[x/2]) - 4*Tan[x/2])/(8*a)
```

**fricas** [B] time = 0.47, size = 134, normalized size = 3.05

$$\frac{8\,\cos(x)^3+6\,\cos(x)^2-3\left(\cos(x)^3+\cos(x)^2+\left(\cos(x)^2-1\right)\sin(x)-\cos(x)-1\right)\log\left(\frac{1}{2}\,\cos(x)+\frac{1}{2}\right)+3\left(\cos(x)\right.}{4\left(a\cos(x)^3+a\cos(x)^2-a\,\mathrm{c}\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^4/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] 1/4*(8*cos(x)^3 + 6*cos(x)^2 - 3*(cos(x)^3 + cos(x)^2 + (cos(x)^2 - 1)*sin(
x) - cos(x) - 1)*log(1/2*cos(x) + 1/2) + 3*(cos(x)^3 + cos(x)^2 + (cos(x)^2
 - 1)*sin(x) - cos(x) - 1)*log(-1/2*cos(x) + 1/2) - 2*(4*cos(x)^2 + cos(x)
- 2)*sin(x) - 6*cos(x) - 4)/(a*cos(x)^3 + a*cos(x)^2 - a*cos(x) + (a*cos(x)
^2 - a)*sin(x) - a)
```

**giac** [A] time = 0.37, size = 73, normalized size = 1.66

$$\frac{3\,\log\left(\left|\tan\left(\frac{1}{2}\,x\right)\right|\right)}{2\,a}+\frac{a\tan\left(\frac{1}{2}\,x\right)^2-4\,a\tan\left(\frac{1}{2}\,x\right)}{8\,a^2}+\frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}-\frac{18\,\tan\left(\frac{1}{2}\,x\right)^2-4\,\tan\left(\frac{1}{2}\,x\right)+1}{8\,a\tan\left(\frac{1}{2}\,x\right)^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^4/(a+a*csc(x)),x, algorithm="giac")
```

```
[Out] 3/2*log(abs(tan(1/2*x)))/a + 1/8*(a*tan(1/2*x)^2 - 4*a*tan(1/2*x))/a^2 + 2/
(a*(tan(1/2*x) + 1)) - 1/8*(18*tan(1/2*x)^2 - 4*tan(1/2*x) + 1)/(a*tan(1/2*
x)^2)
```

**maple** [A] time = 0.27, size = 67, normalized size = 1.52

$$\frac{\tan^2\left(\frac{x}{2}\right)}{8a}-\frac{\tan\left(\frac{x}{2}\right)}{2a}-\frac{1}{8a\tan\left(\frac{x}{2}\right)^2}+\frac{1}{2a\tan\left(\frac{x}{2}\right)}+\frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}+\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^4/(a+a*csc(x)),x)
```

```
[Out] 1/8/a*tan(1/2*x)^2-1/2/a*tan(1/2*x)-1/8/a/tan(1/2*x)^2+1/2/a/tan(1/2*x)+3/2
/a*ln(tan(1/2*x))+2/a/(tan(1/2*x)+1)
```

**maxima [B]** time = 0.33, size = 97, normalized size = 2.20

$$-\frac{\frac{4\sin(x)}{\cos(x)+1}-\frac{\sin(x)^2}{(\cos(x)+1)^2}}{8a}+\frac{\frac{3\sin(x)}{\cos(x)+1}+\frac{20\sin(x)^2}{(\cos(x)+1)^2}-1}{8\left(\frac{a\sin(x)^2}{(\cos(x)+1)^2}+\frac{a\sin(x)^3}{(\cos(x)+1)^3}\right)}+\frac{3\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{2a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^4/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] -1/8*(4*sin(x)/(cos(x) + 1) - sin(x)^2/(cos(x) + 1)^2)/a + 1/8*(3*sin(x)/(c
os(x) + 1) + 20*sin(x)^2/(cos(x) + 1)^2 - 1)/(a*sin(x)^2/(cos(x) + 1)^2 + a
*sin(x)^3/(cos(x) + 1)^3) + 3/2*log(sin(x)/(cos(x) + 1))/a
```

**mupad [B]** time = 0.26, size = 69, normalized size = 1.57

$$\frac{10\tan\left(\frac{x}{2}\right)^2+\frac{3\tan\left(\frac{x}{2}\right)}{2}-\frac{1}{2}}{4a\tan\left(\frac{x}{2}\right)^3+4a\tan\left(\frac{x}{2}\right)^2}-\frac{\tan\left(\frac{x}{2}\right)}{2a}+\frac{\tan\left(\frac{x}{2}\right)^2}{8a}+\frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^4*(a + a/sin(x))),x)
```

```
[Out] ((3*tan(x/2))/2 + 10*tan(x/2)^2 - 1/2)/(4*a*tan(x/2)^2 + 4*a*tan(x/2)^3) -
tan(x/2)/(2*a) + tan(x/2)^2/(8*a) + (3*log(tan(x/2)))/(2*a)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{\csc^4(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**4/(a+a*csc(x)),x)
```

```
[Out] Integral(csc(x)**4/(csc(x) + 1), x)/a
```

## 3.3 $\int \frac{\csc^3(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=27

$$-\frac{\cot(x)}{a}+\frac{\tanh^{-1}(\cos(x))}{a}-\frac{\cot(x)}{a\csc(x)+a}$$

```
[Out] arctanh(cos(x))/a-cot(x)/a-cot(x)/(a+a*csc(x))
```

**Rubi [A]** time = 0.09, antiderivative size = 27, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.308, Rules used = {3790, 3789, 3770, 3794}

$$-\frac{\cot(x)}{a}+\frac{\tanh^{-1}(\cos(x))}{a}-\frac{\cot(x)}{a\csc(x)+a}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^3/(a + a*Csc[x]),x]
```

```
[Out] ArcTanh[Cos[x]]/a - Cot[x]/a - Cot[x]/(a + a*Csc[x])
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3789

```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Dist[1/b, Int[Csc[e + f*x], x], x] - Dist[a/b, Int[Csc[e + f*x]/(a
+ b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

Rule 3790

```
Int[csc[(e_.) + (f_.)*(x_)]^3/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> -Simp[Cot[e + f*x]/(b*f), x] - Dist[a/b, Int[Csc[e + f*x]^2/(a + b*
Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

Rule 3794

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> -Simp[Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\begin{aligned}
\int \frac{\csc^3(x)}{a+a\csc(x)}\,dx &= -\frac{\cot(x)}{a}-\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx\\
&= -\frac{\cot(x)}{a}-\frac{\int \csc(x)\,dx}{a}+\int \frac{\csc(x)}{a+a\csc(x)}\,dx\\
&= \frac{\tanh^{-1}(\cos(x))}{a}-\frac{\cot(x)}{a}-\frac{\cot(x)}{a+a\csc(x)}
\end{aligned}$$

**Mathematica [B]** time = 0.16, size = 63, normalized size = 2.33

$$\frac{\tan\left(\frac{x}{2}\right)-\cot\left(\frac{x}{2}\right)-2\log\left(\sin\left(\frac{x}{2}\right)\right)+2\log\left(\cos\left(\frac{x}{2}\right)\right)+\frac{4\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}}{2a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^3/(a + a*Csc[x]),x]
```

```
[Out] (-Cot[x/2] + 2*Log[Cos[x/2]] - 2*Log[Sin[x/2]] + (4*Sin[x/2])/(Cos[x/2] + S
in[x/2]) + Tan[x/2])/(2*a)
```

**fricas [B]** time = 0.52, size = 91, normalized size = 3.37

$$\frac{4\cos(x)^2+\left(\cos(x)^2-(\cos(x)+1)\sin(x)-1\right)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)-\left(\cos(x)^2-(\cos(x)+1)\sin(x)-1\right)\log\left(-\right.}{2\left(a\cos(x)^2-(a\cos(x)+a)\sin(x)-a\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^3/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] 1/2*(4*cos(x)^2 + (cos(x)^2 - (cos(x) + 1)*sin(x) - 1)*log(1/2*cos(x) + 1/2
) - (cos(x)^2 - (cos(x) + 1)*sin(x) - 1)*log(-1/2*cos(x) + 1/2) + 2*(2*cos(
x) + 1)*sin(x) + 2*cos(x) - 2)/(a*cos(x)^2 - (a*cos(x) + a)*sin(x) - a)
```

**giac [A]** time = 0.56, size = 53, normalized size = 1.96

$$-\frac{\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{a}+\frac{\tan\left(\frac{1}{2}x\right)}{2a}+\frac{\tan\left(\frac{1}{2}x\right)^2-4\tan\left(\frac{1}{2}x\right)-1}{2\left(\tan\left(\frac{1}{2}x\right)^2+\tan\left(\frac{1}{2}x\right)\right)a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^3/(a+a*csc(x)),x, algorithm="giac")
```

```
[Out] -log(abs(tan(1/2*x)))/a + 1/2*tan(1/2*x)/a + 1/2*(tan(1/2*x)^2 - 4*tan(1/2*
x) - 1)/((tan(1/2*x)^2 + tan(1/2*x))*a)
```

**maple [A]** time = 0.23, size = 45, normalized size = 1.67

$$\frac{\tan\left(\frac{x}{2}\right)}{2a}-\frac{1}{2a\tan\left(\frac{x}{2}\right)}-\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^3/(a+a*csc(x)),x)
```

```
[Out] 1/2/a*tan(1/2*x)-1/2/a/tan(1/2*x)-1/a*ln(tan(1/2*x))-2/a/(tan(1/2*x)+1)
```

**maxima [B]** time = 0.33, size = 68, normalized size = 2.52

$$-\frac{\frac{5\sin(x)}{\cos(x)+1}+1}{2\left(\frac{a\sin(x)}{\cos(x)+1}+\frac{a\sin(x)^2}{(\cos(x)+1)^2}\right)}-\frac{\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}+\frac{\sin(x)}{2a(\cos(x)+1)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^3/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] -1/2*(5*sin(x)/(cos(x) + 1) + 1)/(a*sin(x)/(cos(x) + 1) + a*sin(x)^2/(cos(x
) + 1)^2) - log(sin(x)/(cos(x) + 1))/a + 1/2*sin(x)/(a*(cos(x) + 1))
```

**mupad [B]** time = 0.25, size = 49, normalized size = 1.81

$$\frac{\tan\left(\frac{x}{2}\right)}{2\,a}-\frac{5\tan\left(\frac{x}{2}\right)+1}{2\,a\tan\left(\frac{x}{2}\right)^2+2\,a\tan\left(\frac{x}{2}\right)}-\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^3*(a + a/sin(x))),x)
```

```
[Out] tan(x/2)/(2*a) - (5*tan(x/2) + 1)/(2*a*tan(x/2) + 2*a*tan(x/2)^2) - log(tan
(x/2))/a
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{\csc^3(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**3/(a+a*csc(x)),x)
```

```
[Out] Integral(csc(x)**3/(csc(x) + 1), x)/a
```

## 3.4 $\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=20

$$\frac{\cot(x)}{a\csc(x)+a}-\frac{\tanh^{-1}(\cos(x))}{a}$$

```
[Out] -arctanh(cos(x))/a+cot(x)/(a+a*csc(x))
```

**Rubi [A]** time = 0.06, antiderivative size = 20, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.231, Rules used = {3789, 3770, 3794}

$$\frac{\cot(x)}{a\csc(x)+a}-\frac{\tanh^{-1}(\cos(x))}{a}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^2/(a + a*Csc[x]),x]
```

```
[Out] -(ArcTanh[Cos[x]]/a) + Cot[x]/(a + a*Csc[x])
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3789

```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Dist[1/b, Int[Csc[e + f*x], x], x] - Dist[a/b, Int[Csc[e + f*x]/(a
+ b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

Rule 3794

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> -Simp[Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx &= \frac{\int \csc(x)\,dx}{a}-\int \frac{\csc(x)}{a+a\csc(x)}\,dx\\ &= -\frac{\tanh^{-1}(\cos(x))}{a}+\frac{\cot(x)}{a+a\csc(x)}\end{aligned}$$

**Mathematica [B]** time = 0.05, size = 44, normalized size = 2.20

$$\frac{\log\left(\sin\left(\frac{x}{2}\right)\right)-\log\left(\cos\left(\frac{x}{2}\right)\right)-\frac{2\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}}{a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^2/(a + a*Csc[x]),x]
```

```
[Out] (-Log[Cos[x/2]] + Log[Sin[x/2]] - (2*Sin[x/2])/(Cos[x/2] + Sin[x/2]))/a
```

**fricas** [B] time = 0.49, size = 53, normalized size = 2.65

$$-\frac{(\cos(x)+\sin(x)+1)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)-(\cos(x)+\sin(x)+1)\log\left(-\frac{1}{2}\cos(x)+\frac{1}{2}\right)-2\cos(x)+2\sin(x)-2}{2\,(a\cos(x)+a\sin(x)+a)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^2/(a+a*csc(x)),x, algorithm="fricas")
[Out] -1/2*((cos(x) + sin(x) + 1)*log(1/2*cos(x) + 1/2) - (cos(x) + sin(x) + 1)*l
og(-1/2*cos(x) + 1/2) - 2*cos(x) + 2*sin(x) - 2)/(a*cos(x) + a*sin(x) + a)
```

**giac** [A] time = 0.60, size = 24, normalized size = 1.20

$$\frac{\log\left(\left|\tan\left(\frac{1}{2}\,x\right)\right|\right)}{a}+\frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^2/(a+a*csc(x)),x, algorithm="giac")
[Out] log(abs(tan(1/2*x)))/a + 2/(a*(tan(1/2*x) + 1))
```

**maple** [A] time = 0.25, size = 24, normalized size = 1.20

$$\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^2/(a+a*csc(x)),x)
[Out] 2/a/(tan(1/2*x)+1)+1/a*ln(tan(1/2*x))
```

**maxima** [A] time = 0.32, size = 31, normalized size = 1.55

$$\frac{\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}+\frac{2}{a+\frac{a\sin(x)}{\cos(x)+1}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^2/(a+a*csc(x)),x, algorithm="maxima")
[Out] log(sin(x)/(cos(x) + 1))/a + 2/(a + a*sin(x)/(cos(x) + 1))
```

**mupad** [B] time = 0.18, size = 23, normalized size = 1.15

$$\frac{2}{a\,\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^2*(a + a/sin(x))),x)
[Out] 2/(a*(tan(x/2) + 1)) + log(tan(x/2))/a
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int\frac{\csc^2(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**2/(a+a*csc(x)),x)
```

```
[Out] Integral(csc(x)**2/(csc(x) + 1), x)/a
```

## 3.5 $\int \frac{\csc(x)}{a+a\csc(x)}\, dx$

**Optimal**. Leaf size=12

$$-\frac{\cot(x)}{a\csc(x)+a}$$

```
[Out] -cot(x)/(a+a*csc(x))
```

**Rubi [A]** time = 0.02, antiderivative size = 12, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 1, integrand size = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.091, Rules used = {3794}

$$-\frac{\cot(x)}{a\csc(x)+a}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]/(a + a*Csc[x]),x]
```

```
[Out] -(Cot[x]/(a + a*Csc[x]))
```

Rule 3794

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> -Simp[Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{\csc(x)}{a+a\csc(x)}\, dx = -\frac{\cot(x)}{a+a\csc(x)}$$

**Mathematica [B]** time = 0.02, size = 26, normalized size = 2.17

$$\frac{2\sin\left(\frac{x}{2}\right)}{a\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]/(a + a*Csc[x]),x]
```

```
[Out] (2*Sin[x/2])/(a*(Cos[x/2] + Sin[x/2]))
```

**fricas [A]** time = 0.50, size = 22, normalized size = 1.83

$$-\frac{\cos(x)-\sin(x)+1}{a\cos(x)+a\sin(x)+a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] -(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a)
```

**giac [A]** time = 0.45, size = 13, normalized size = 1.08

$$-\frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+a*csc(x)),x, algorithm="giac")
[Out] -2/(a*(tan(1/2*x) + 1))
```

**maple [A]** time = 0.20, size = 14, normalized size = 1.17

$$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)/(a+a*csc(x)),x)
[Out] -2/a/(tan(1/2*x)+1)
```

**maxima [A]** time = 0.34, size = 16, normalized size = 1.33

$$-\frac{2}{a+\frac{a\sin(x)}{\cos(x)+1}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+a*csc(x)),x, algorithm="maxima")
[Out] -2/(a + a*sin(x)/(cos(x) + 1))
```

**mupad [B]** time = 0.17, size = 13, normalized size = 1.08

$$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)*(a + a/sin(x))),x)
[Out] -2/(a*(tan(x/2) + 1))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int\frac{\csc(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+a*csc(x)),x)
[Out] Integral(csc(x)/(csc(x) + 1), x)/a
```

## 3.6 $\int \frac{1}{a+a\csc(c+dx)}\,dx$

**Optimal**. Leaf size=28

$$\frac{\cot(c+dx)}{d(a\csc(c+dx)+a)}+\frac{x}{a}$$

```
[Out] x/a+cot(d*x+c)/d/(a+a*csc(d*x+c))
```

**Rubi [A]** time = 0.01, antiderivative size = 28, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.167, Rules used = {3777, 8}

$$\frac{\cot(c+dx)}{d(a\csc(c+dx)+a)}+\frac{x}{a}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[c + d*x])^(-1),x]
```

```
[Out] x/a + Cot[c + d*x]/(d*(a + a*Csc[c + d*x]))
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 3777

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(Cot[c
+ d*x]*(a + b*Csc[c + d*x])^n)/(d*(2*n + 1)), x] + Dist[1/(a^2*(2*n + 1)),
Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Intege
rQ[2*n]
```

Rubi steps

$$\begin{aligned}\int \frac{1}{a+a\csc(c+dx)}\,dx &= \frac{\cot(c+dx)}{d(a+a\csc(c+dx))}+\frac{\int a\,dx}{a^2}\\ &= \frac{x}{a}+\frac{\cot(c+dx)}{d(a+a\csc(c+dx))}\end{aligned}$$

**Mathematica [A]** time = 0.09, size = 47, normalized size = 1.68

$$\frac{-\frac{2\sin\left(\frac{1}{2}(c+dx)\right)}{\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)}+c+dx}{ad}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[c + d*x])^(-1),x]
```

```
[Out] (c + d*x - (2*Sin[(c + d*x)/2])/(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))/(a*d
)
```

**fricas [A]** time = 0.56, size = 54, normalized size = 1.93

$$\frac{dx+(dx+1)\cos(dx+c)+(dx-1)\sin(dx+c)+1}{ad\cos(dx+c)+ad\sin(dx+c)+ad}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c)),x, algorithm="fricas")
```

```
[Out] (d*x + (d*x + 1)*cos(d*x + c) + (d*x - 1)*sin(d*x + c) + 1)/(a*d*cos(d*x +
c) + a*d*sin(d*x + c) + a*d)
```

**giac [A]** time = 0.42, size = 32, normalized size = 1.14

$$\frac{\frac{dx+c}{a} + \frac{2}{a\left(\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+1\right)}}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] ((d*x + c)/a + 2/(a*(tan(1/2*d*x + 1/2*c) + 1)))/d
```

**maple [A]** time = 0.52, size = 41, normalized size = 1.46

$$\frac{2}{ad\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{ad}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+a*csc(d*x+c)),x)
```

```
[Out] 2/a/d/(tan(1/2*d*x+1/2*c)+1)+2/a/d*arctan(tan(1/2*d*x+1/2*c))
```

**maxima [A]** time = 0.42, size = 50, normalized size = 1.79

$$\frac{2\left(\frac{\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{a}+\frac{1}{a+\frac{a\sin(dx+c)}{\cos(dx+c)+1}}\right)}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c)),x, algorithm="maxima")
```

```
[Out] 2*(arctan(sin(d*x + c)/(cos(d*x + c) + 1))/a + 1/(a + a*sin(d*x + c)/(cos(d
*x + c) + 1)))/d
```

**mupad [B]** time = 0.22, size = 27, normalized size = 0.96

$$\frac{x}{a}+\frac{2}{a\,d\,\left(\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + a/sin(c + d*x)),x)
```

```
[Out] x/a + 2/(a*d*(tan(c/2 + (d*x)/2) + 1))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{1}{\csc(c+dx)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c)),x)
```

```
[Out] Integral(1/(csc(c + d*x) + 1), x)/a
```

## 3.7 $\int \frac{\sin(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=25

$$-\frac{x}{a}-\frac{2\cos(x)}{a}+\frac{\cos(x)}{a\csc(x)+a}$$

```
[Out] -x/a-2*cos(x)/a+cos(x)/(a+a*csc(x))
```

**Rubi [A]** time = 0.05, antiderivative size = 25, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.364, Rules used = {3819, 3787, 2638, 8}

$$-\frac{x}{a}-\frac{2\cos(x)}{a}+\frac{\cos(x)}{a\csc(x)+a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]/(a + a*Csc[x]),x]
```

```
[Out] -(x/a) - (2*Cos[x])/a + Cos[x]/(a + a*Csc[x])
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 2638

```
Int[sin[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[Cos[c + d*x]/d, x] /; FreeQ
[{c, d}, x]
```

Rule 3787

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Dist[a, Int[(d*Csc[e + f*x])^n, x], x] + Dist[b/d, Int[
(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

Rule 3819

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(f*(a + b*Csc[e +
 f*x])), x] - Dist[1/a^2, Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e + f
*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{\sin(x)}{a+a\csc(x)}\,dx &= \frac{\cos(x)}{a+a\csc(x)}-\frac{\int(-2a+a\csc(x))\sin(x)\,dx}{a^2}\\ &= \frac{\cos(x)}{a+a\csc(x)}-\frac{\int 1\,dx}{a}+\frac{2\int\sin(x)\,dx}{a}\\ &= -\frac{x}{a}-\frac{2\cos(x)}{a}+\frac{\cos(x)}{a+a\csc(x)}\end{aligned}$$

**Mathematica [A]** time = 0.08, size = 32, normalized size = 1.28

$$-\frac{x+\cos(x)-\frac{2\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}}{a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]/(a + a*Csc[x]),x]
```

```
[Out] -((x + Cos[x] - (2*Sin[x/2])/(Cos[x/2] + Sin[x/2]))/a)
```

**fricas [A]** time = 0.61, size = 35, normalized size = 1.40

$$-\frac{(x+2)\cos(x)+\cos(x)^2+(x+\cos(x)-1)\sin(x)+x+1}{a\cos(x)+a\sin(x)+a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] -((x + 2)*cos(x) + cos(x)^2 + (x + cos(x) - 1)*sin(x) + x + 1)/(a*cos(x) +
a*sin(x) + a)
```

**giac [A]** time = 0.53, size = 44, normalized size = 1.76

$$-\frac{x}{a}-\frac{2\left(\tan\left(\frac{1}{2}x\right)^2+\tan\left(\frac{1}{2}x\right)+2\right)}{\left(\tan\left(\frac{1}{2}x\right)^3+\tan\left(\frac{1}{2}x\right)^2+\tan\left(\frac{1}{2}x\right)+1\right)a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+a*csc(x)),x, algorithm="giac")
```

```
[Out] -x/a - 2*(tan(1/2*x)^2 + tan(1/2*x) + 2)/((tan(1/2*x)^3 + tan(1/2*x)^2 + ta
n(1/2*x) + 1)*a)
```

**maple [A]** time = 0.40, size = 40, normalized size = 1.60

$$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}-\frac{2}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)}-\frac{2\arctan\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)/(a+a*csc(x)),x)
```

```
[Out] -2/a/(tan(1/2*x)+1)-2/a/(tan(1/2*x)^2+1)-2/a*arctan(tan(1/2*x))
```

**maxima [B]** time = 0.43, size = 78, normalized size = 3.12

$$-\frac{2\left(\frac{\sin(x)}{\cos(x)+1}+\frac{\sin(x)^2}{(\cos(x)+1)^2}+2\right)}{a+\frac{a\sin(x)}{\cos(x)+1}+\frac{a\sin(x)^2}{(\cos(x)+1)^2}+\frac{a\sin(x)^3}{(\cos(x)+1)^3}}-\frac{2\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] -2*(sin(x)/(cos(x) + 1) + sin(x)^2/(cos(x) + 1)^2 + 2)/(a + a*sin(x)/(cos(x
) + 1) + a*sin(x)^2/(cos(x) + 1)^2 + a*sin(x)^3/(cos(x) + 1)^3) - 2*arctan(
sin(x)/(cos(x) + 1))/a
```

**mupad [B]** time = 0.24, size = 46, normalized size = 1.84

$$-\frac{x}{a}-\frac{2\tan\left(\frac{x}{2}\right)^2+2\tan\left(\frac{x}{2}\right)+4}{a\left(\tan\left(\frac{x}{2}\right)^2+1\right)\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)/(a + a/sin(x)),x)
[Out] - x/a - (2*tan(x/2) + 2*tan(x/2)^2 + 4)/(a*(tan(x/2)^2 + 1)*(tan(x/2) + 1))
```

**sympy** **[F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{\sin(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+a*csc(x)),x)
[Out] Integral(sin(x)/(csc(x) + 1), x)/a
```

## 3.8 $\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=40

$$\frac{3x}{2a}+\frac{2\cos(x)}{a}-\frac{3\sin(x)\cos(x)}{2a}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

```
[Out] 3/2*x/a+2*cos(x)/a-3/2*cos(x)*sin(x)/a+cos(x)*sin(x)/(a+a*csc(x))
```

**Rubi** [A] time = 0.06, antiderivative size = 40, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {3819, 3787, 2635, 8, 2638}

$$\frac{3x}{2a}+\frac{2\cos(x)}{a}-\frac{3\sin(x)\cos(x)}{2a}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]^2/(a + a*Csc[x]),x]
```

```
[Out] (3*x)/(2*a) + (2*Cos[x])/a - (3*Cos[x]*Sin[x])/(2*a) + (Cos[x]*Sin[x])/(a +
 a*Csc[x])
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 2635

```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> -Simp[(b*Cos[c + d*x
]*(b*Sin[c + d*x])^(n - 1))/(d*n), x] + Dist[(b^2*(n - 1))/n, Int[(b*Sin[c
+ d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n
]
```

Rule 2638

```
Int[sin[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[Cos[c + d*x]/d, x] /; FreeQ
[{c, d}, x]
```

Rule 3787

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Dist[a, Int[(d*Csc[e + f*x])^n, x], x] + Dist[b/d, Int[
(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

Rule 3819

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(f*(a + b*Csc[e +
 f*x])), x] - Dist[1/a^2, Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e + f
*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx &= \frac{\cos(x)\sin(x)}{a+a\csc(x)} - \frac{\int(-3a+2a\csc(x))\sin^2(x)\,dx}{a^2}\\ &= \frac{\cos(x)\sin(x)}{a+a\csc(x)} - \frac{2\int\sin(x)\,dx}{a} + \frac{3\int\sin^2(x)\,dx}{a}\\ &= \frac{2\cos(x)}{a} - \frac{3\cos(x)\sin(x)}{2a} + \frac{\cos(x)\sin(x)}{a+a\csc(x)} + \frac{3\int 1\,dx}{2a}\\ &= \frac{3x}{2a} + \frac{2\cos(x)}{a} - \frac{3\cos(x)\sin(x)}{2a} + \frac{\cos(x)\sin(x)}{a+a\csc(x)}\end{aligned}$$

**Mathematica [A]** time = 0.13, size = 42, normalized size = 1.05

$$-\frac{-6x+\sin(2x)-4\cos(x)+\frac{8\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}}{4a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]^2/(a + a*Csc[x]),x]
```

```
[Out] -1/4*(-6*x - 4*Cos[x] + (8*Sin[x/2])/(Cos[x/2] + Sin[x/2]) + Sin[2*x])/a
```

**fricas [A]** time = 0.52, size = 53, normalized size = 1.32

$$\frac{\cos(x)^3+3\,(x+1)\cos(x)+2\,\cos(x)^2-\left(\cos(x)^2-3\,x-\cos(x)+2\right)\sin(x)+3\,x+2}{2\,(a\cos(x)+a\sin(x)+a)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^2/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] 1/2*(cos(x)^3 + 3*(x + 1)*cos(x) + 2*cos(x)^2 - (cos(x)^2 - 3*x - cos(x) +
2)*sin(x) + 3*x + 2)/(a*cos(x) + a*sin(x) + a)
```

**giac [A]** time = 0.69, size = 56, normalized size = 1.40

$$\frac{3\,x}{2\,a}+\frac{\tan\left(\frac{1}{2}\,x\right)^3+2\,\tan\left(\frac{1}{2}\,x\right)^2-\tan\left(\frac{1}{2}\,x\right)+2}{\left(\tan\left(\frac{1}{2}\,x\right)^2+1\right)^2 a}+\frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^2/(a+a*csc(x)),x, algorithm="giac")
```

```
[Out] 3/2*x/a + (tan(1/2*x)^3 + 2*tan(1/2*x)^2 - tan(1/2*x) + 2)/((tan(1/2*x)^2 +
 1)^2*a) + 2/(a*(tan(1/2*x) + 1))
```

**maple [B]** time = 0.52, size = 100, normalized size = 2.50

$$\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{\tan^3\left(\frac{x}{2}\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}+\frac{2\left(\tan^2\left(\frac{x}{2}\right)\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}-\frac{\tan\left(\frac{x}{2}\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}+\frac{2}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}+\frac{3\arctan\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^2/(a+a*csc(x)),x)
```

```
[Out] 2/a/(tan(1/2*x)+1)+1/a/(tan(1/2*x)^2+1)^2*tan(1/2*x)^3+2/a/(tan(1/2*x)^2+1)
^2*tan(1/2*x)^2-1/a/(tan(1/2*x)^2+1)^2*tan(1/2*x)+2/a/(tan(1/2*x)^2+1)^2+3/
a*arctan(tan(1/2*x))
```

**maxima [B]** time = 0.43, size = 128, normalized size = 3.20

$$\frac{\frac{\sin(x)}{\cos(x)+1} + \frac{5\sin(x)^2}{(\cos(x)+1)^2} + \frac{3\sin(x)^3}{(\cos(x)+1)^3} + \frac{3\sin(x)^4}{(\cos(x)+1)^4} + 4}{a + \frac{a\sin(x)}{\cos(x)+1} + \frac{2a\sin(x)^2}{(\cos(x)+1)^2} + \frac{2a\sin(x)^3}{(\cos(x)+1)^3} + \frac{a\sin(x)^4}{(\cos(x)+1)^4} + \frac{a\sin(x)^5}{(\cos(x)+1)^5}} + \frac{3\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^2/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] (sin(x)/(cos(x) + 1) + 5*sin(x)^2/(cos(x) + 1)^2 + 3*sin(x)^3/(cos(x) + 1)^
3 + 3*sin(x)^4/(cos(x) + 1)^4 + 4)/(a + a*sin(x)/(cos(x) + 1) + 2*a*sin(x)^
2/(cos(x) + 1)^2 + 2*a*sin(x)^3/(cos(x) + 1)^3 + a*sin(x)^4/(cos(x) + 1)^4
+ a*sin(x)^5/(cos(x) + 1)^5) + 3*arctan(sin(x)/(cos(x) + 1))/a
```

**mupad [B]** time = 0.27, size = 59, normalized size = 1.48

$$\frac{3x}{2a} + \frac{3\tan\left(\frac{x}{2}\right)^4 + 3\tan\left(\frac{x}{2}\right)^3 + 5\tan\left(\frac{x}{2}\right)^2 + \tan\left(\frac{x}{2}\right) + 4}{a\left(\tan\left(\frac{x}{2}\right)^2 + 1\right)^2\left(\tan\left(\frac{x}{2}\right) + 1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^2/(a + a/sin(x)),x)
```

```
[Out] (3*x)/(2*a) + (tan(x/2) + 5*tan(x/2)^2 + 3*tan(x/2)^3 + 3*tan(x/2)^4 + 4)/(
a*(tan(x/2)^2 + 1)^2*(tan(x/2) + 1))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{\sin^2(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)**2/(a+a*csc(x)),x)
```

```
[Out] Integral(sin(x)**2/(csc(x) + 1), x)/a
```

## 3.9 $\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=53

$$-\frac{3x}{2a}+\frac{4\cos^3(x)}{3a}-\frac{4\cos(x)}{a}+\frac{3\sin(x)\cos(x)}{2a}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

```
[Out] -3/2*x/a-4*cos(x)/a+4/3*cos(x)^3/a+3/2*cos(x)*sin(x)/a+cos(x)*sin(x)^2/(a+a
*csc(x))
```

**Rubi [A]** time = 0.07, antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {3819, 3787, 2633, 2635, 8}

$$-\frac{3x}{2a}+\frac{4\cos^3(x)}{3a}-\frac{4\cos(x)}{a}+\frac{3\sin(x)\cos(x)}{2a}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]^3/(a + a*Csc[x]),x]
```

```
[Out] (-3*x)/(2*a) - (4*Cos[x])/a + (4*Cos[x]^3)/(3*a) + (3*Cos[x]*Sin[x])/(2*a)
+ (Cos[x]*Sin[x]^2)/(a + a*Csc[x])
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 2633

```
Int[sin[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
nd[(1 - x^2)^((n - 1)/2), x], x], x, Cos[c + d*x]], x] /; FreeQ[{c, d}, x]
&& IGtQ[(n - 1)/2, 0]
```

Rule 2635

```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> -Simp[(b*Cos[c + d*x
]*(b*Sin[c + d*x])^(n - 1))/(d*n), x] + Dist[(b^2*(n - 1))/n, Int[(b*Sin[c
+ d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n
]
```

Rule 3787

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Dist[a, Int[(d*Csc[e + f*x])^n, x], x] + Dist[b/d, Int[
(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

Rule 3819

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(f*(a + b*Csc[e +
 f*x])), x] - Dist[1/a^2, Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e + f
*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx &= \frac{\cos(x)\sin^2(x)}{a+a\csc(x)}-\frac{\int(-4a+3a\csc(x))\sin^3(x)\,dx}{a^2}\\ &= \frac{\cos(x)\sin^2(x)}{a+a\csc(x)}-\frac{3\int\sin^2(x)\,dx}{a}+\frac{4\int\sin^3(x)\,dx}{a}\\ &= \frac{3\cos(x)\sin(x)}{2a}+\frac{\cos(x)\sin^2(x)}{a+a\csc(x)}-\frac{3\int 1\,dx}{2a}-\frac{4\,\mathrm{Subst}\left(\int\left(1-x^2\right)dx,x,\cos(x)\right)}{a}\\ &= -\frac{3x}{2a}-\frac{4\cos(x)}{a}+\frac{4\cos^3(x)}{3a}+\frac{3\cos(x)\sin(x)}{2a}+\frac{\cos(x)\sin^2(x)}{a+a\csc(x)}\end{aligned}$$

**Mathematica** [A] time = 0.17, size = 49, normalized size = 0.92

$$\frac{-21\cos(x)+\cos(3x)+3\left(-6x+\sin(2x)+\frac{8\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}\right)}{12a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]^3/(a + a*Csc[x]),x]
```

```
[Out] (-21*Cos[x] + Cos[3*x] + 3*(-6*x + (8*Sin[x/2])/(Cos[x/2] + Sin[x/2]) + Sin
[2*x]))/(12*a)
```

**fricas** [A] time = 0.45, size = 70, normalized size = 1.32

$$\frac{2\,\cos(x)^4-\cos(x)^3-3\,(3\,x+5)\cos(x)-12\,\cos(x)^2+\left(2\,\cos(x)^3+3\,\cos(x)^2-9\,x-9\,\cos(x)+6\right)\sin(x)-9}{6\,(a\cos(x)+a\sin(x)+a)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^3/(a+a*csc(x)),x, algorithm="fricas")
```

```
[Out] 1/6*(2*cos(x)^4 - cos(x)^3 - 3*(3*x + 5)*cos(x) - 12*cos(x)^2 + (2*cos(x)^3
 + 3*cos(x)^2 - 9*x - 9*cos(x) + 6)*sin(x) - 9*x - 6)/(a*cos(x) + a*sin(x)
+ a)
```

**giac** [A] time = 0.29, size = 67, normalized size = 1.26

$$-\frac{3\,x}{2\,a}-\frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}-\frac{3\,\tan\left(\frac{1}{2}\,x\right)^5+6\,\tan\left(\frac{1}{2}\,x\right)^4+24\,\tan\left(\frac{1}{2}\,x\right)^2-3\,\tan\left(\frac{1}{2}\,x\right)+10}{3\left(\tan\left(\frac{1}{2}\,x\right)^2+1\right)^3a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^3/(a+a*csc(x)),x, algorithm="giac")
```

```
[Out] -3/2*x/a - 2/(a*(tan(1/2*x) + 1)) - 1/3*(3*tan(1/2*x)^5 + 6*tan(1/2*x)^4 +
24*tan(1/2*x)^2 - 3*tan(1/2*x) + 10)/((tan(1/2*x)^2 + 1)^3*a)
```

**maple** [B] time = 0.51, size = 121, normalized size = 2.28

$$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}-\frac{\tan^5\left(\frac{x}{2}\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}-\frac{2\left(\tan^4\left(\frac{x}{2}\right)\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}-\frac{8\left(\tan^2\left(\frac{x}{2}\right)\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}+\frac{\tan\left(\frac{x}{2}\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}-\frac{10}{3a\left(\tan^2\left(\frac{x}{2}\right)+1\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^3/(a+a*csc(x)),x)
```

```
[Out] -2/a/(tan(1/2*x)+1)-1/a/(tan(1/2*x)^2+1)^3*tan(1/2*x)^5-2/a/(tan(1/2*x)^2+1
)^3*tan(1/2*x)^4-8/a/(tan(1/2*x)^2+1)^3*tan(1/2*x)^2+1/a/(tan(1/2*x)^2+1)^3
*tan(1/2*x)-10/3/a/(tan(1/2*x)^2+1)^3-3/a*arctan(tan(1/2*x))
```

**maxima [B]** time = 0.43, size = 180, normalized size = 3.40

$$-\frac{\frac{7\sin(x)}{\cos(x)+1}+\frac{39\sin(x)^2}{(\cos(x)+1)^2}+\frac{24\sin(x)^3}{(\cos(x)+1)^3}+\frac{24\sin(x)^4}{(\cos(x)+1)^4}+\frac{9\sin(x)^5}{(\cos(x)+1)^5}+\frac{9\sin(x)^6}{(\cos(x)+1)^6}+16}{3\left(a+\frac{a\sin(x)}{\cos(x)+1}+\frac{3a\sin(x)^2}{(\cos(x)+1)^2}+\frac{3a\sin(x)^3}{(\cos(x)+1)^3}+\frac{3a\sin(x)^4}{(\cos(x)+1)^4}+\frac{3a\sin(x)^5}{(\cos(x)+1)^5}+\frac{a\sin(x)^6}{(\cos(x)+1)^6}+\frac{a\sin(x)^7}{(\cos(x)+1)^7}\right)}-\frac{3\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^3/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] -1/3*(7*sin(x)/(cos(x) + 1) + 39*sin(x)^2/(cos(x) + 1)^2 + 24*sin(x)^3/(cos
(x) + 1)^3 + 24*sin(x)^4/(cos(x) + 1)^4 + 9*sin(x)^5/(cos(x) + 1)^5 + 9*sin
(x)^6/(cos(x) + 1)^6 + 16)/(a + a*sin(x)/(cos(x) + 1) + 3*a*sin(x)^2/(cos(x
) + 1)^2 + 3*a*sin(x)^3/(cos(x) + 1)^3 + 3*a*sin(x)^4/(cos(x) + 1)^4 + 3*a*
sin(x)^5/(cos(x) + 1)^5 + a*sin(x)^6/(cos(x) + 1)^6 + a*sin(x)^7/(cos(x) +
1)^7) - 3*arctan(sin(x)/(cos(x) + 1))/a
```

**mupad [B]** time = 0.35, size = 78, normalized size = 1.47

$$-\frac{3x}{2a}-\frac{3\tan\left(\frac{x}{2}\right)^6+3\tan\left(\frac{x}{2}\right)^5+8\tan\left(\frac{x}{2}\right)^4+8\tan\left(\frac{x}{2}\right)^3+13\tan\left(\frac{x}{2}\right)^2+\frac{7\tan\left(\frac{x}{2}\right)}{3}+\frac{16}{3}}{a\left(\tan\left(\frac{x}{2}\right)^2+1\right)^3\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^3/(a + a/sin(x)),x)
```

```
[Out] - (3*x)/(2*a) - ((7*tan(x/2))/3 + 13*tan(x/2)^2 + 8*tan(x/2)^3 + 8*tan(x/2)
^4 + 3*tan(x/2)^5 + 3*tan(x/2)^6 + 16/3)/(a*(tan(x/2)^2 + 1)^3*(tan(x/2) +
1))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int\frac{\sin^3(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)**3/(a+a*csc(x)),x)
```

```
[Out] Integral(sin(x)**3/(csc(x) + 1), x)/a
```

## 3.10 $\int \frac{\sin^4(x)}{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=66

$$\frac{15x}{8a}-\frac{4\cos^3(x)}{3a}+\frac{4\cos(x)}{a}-\frac{5\sin^3(x)\cos(x)}{4a}-\frac{15\sin(x)\cos(x)}{8a}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

```
[Out] 15/8*x/a+4*cos(x)/a-4/3*cos(x)^3/a-15/8*cos(x)*sin(x)/a-5/4*cos(x)*sin(x)^3
/a+cos(x)*sin(x)^3/(a+a*csc(x))
```

**Rubi [A]** time = 0.07, antiderivative size = 66, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 5, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {3819, 3787, 2635, 8, 2633}

$$\frac{15x}{8a}-\frac{4\cos^3(x)}{3a}+\frac{4\cos(x)}{a}-\frac{5\sin^3(x)\cos(x)}{4a}-\frac{15\sin(x)\cos(x)}{8a}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]^4/(a + a*Csc[x]),x]
```

```
[Out] (15*x)/(8*a) + (4*Cos[x])/a - (4*Cos[x]^3)/(3*a) - (15*Cos[x]*Sin[x])/(8*a)
 - (5*Cos[x]*Sin[x]^3)/(4*a) + (Cos[x]*Sin[x]^3)/(a + a*Csc[x])
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 2633

```
Int[sin[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
nd[(1 - x^2)^((n - 1)/2), x], x], x, Cos[c + d*x]], x] /; FreeQ[{c, d}, x]
&& IGtQ[(n - 1)/2, 0]
```

Rule 2635

```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> -Simp[(b*Cos[c + d*x
]*(b*Sin[c + d*x])^(n - 1))/(d*n), x] + Dist[(b^2*(n - 1))/n, Int[(b*Sin[c
+ d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n
]
```

Rule 3787

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Dist[a, Int[(d*Csc[e + f*x])^n, x], x] + Dist[b/d, Int[
(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

Rule 3819

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(f*(a + b*Csc[e +
 f*x])), x] - Dist[1/a^2, Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e + f
*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0]
```

Rubi steps

$$\int \frac{\sin^4(x)}{a+a\csc(x)}\,dx = \frac{\cos(x)\sin^3(x)}{a+a\csc(x)} - \frac{\int(-5a+4a\csc(x))\sin^4(x)\,dx}{a^2}$$
$$= \frac{\cos(x)\sin^3(x)}{a+a\csc(x)} - \frac{4\int\sin^3(x)\,dx}{a} + \frac{5\int\sin^4(x)\,dx}{a}$$
$$= -\frac{5\cos(x)\sin^3(x)}{4a} + \frac{\cos(x)\sin^3(x)}{a+a\csc(x)} + \frac{15\int\sin^2(x)\,dx}{4a} + \frac{4\,\text{Subst}\left(\int\left(1-x^2\right)dx, x, \cos(x)\right)}{a}$$
$$= \frac{4\cos(x)}{a} - \frac{4\cos^3(x)}{3a} - \frac{15\cos(x)\sin(x)}{8a} - \frac{5\cos(x)\sin^3(x)}{4a} + \frac{\cos(x)\sin^3(x)}{a+a\csc(x)} + \frac{15\int 1\,dx}{8a}$$
$$= \frac{15x}{8a} + \frac{4\cos(x)}{a} - \frac{4\cos^3(x)}{3a} - \frac{15\cos(x)\sin(x)}{8a} - \frac{5\cos(x)\sin^3(x)}{4a} + \frac{\cos(x)\sin^3(x)}{a+a\csc(x)}$$

**Mathematica [A]** time = 0.25, size = 57, normalized size = 0.86

$$\frac{168\cos(x) - 8\cos(3x) + 3\left(60x - 16\sin(2x) + \sin(4x) - \frac{64\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}\right)}{96a}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]^4/(a + a*Csc[x]),x]
[Out] (168*Cos[x] - 8*Cos[3*x] + 3*(60*x - (64*Sin[x/2])/(Cos[x/2] + Sin[x/2]) -
16*Sin[2*x] + Sin[4*x]))/(96*a)
```

**fricas [A]** time = 0.59, size = 81, normalized size = 1.23

$$-\frac{6\,\cos(x)^5 + 8\,\cos(x)^4 - 25\,\cos(x)^3 - 45\,(x+1)\cos(x) - 48\,\cos(x)^2 - \left(6\,\cos(x)^4 - 2\,\cos(x)^3 - 27\,\cos(x)^2 + 45\right.}{24\,(a\cos(x) + a\sin(x) + a)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^4/(a+a*csc(x)),x, algorithm="fricas")
[Out] -1/24*(6*cos(x)^5 + 8*cos(x)^4 - 25*cos(x)^3 - 45*(x + 1)*cos(x) - 48*cos(x
)^2 - (6*cos(x)^4 - 2*cos(x)^3 - 27*cos(x)^2 + 45*x + 21*cos(x) - 24)*sin(x
) - 45*x - 24)/(a*cos(x) + a*sin(x) + a)
```

**giac [A]** time = 0.34, size = 91, normalized size = 1.38

$$\frac{15\,x}{8\,a} + \frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right)+1\right)} + \frac{21\,\tan\left(\frac{1}{2}\,x\right)^7 + 24\,\tan\left(\frac{1}{2}\,x\right)^6 + 45\,\tan\left(\frac{1}{2}\,x\right)^5 + 120\,\tan\left(\frac{1}{2}\,x\right)^4 - 45\,\tan\left(\frac{1}{2}\,x\right)^3 + 136\,\text{ta}}{12\left(\tan\left(\frac{1}{2}\,x\right)^2+1\right)^4 a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^4/(a+a*csc(x)),x, algorithm="giac")
[Out] 15/8*x/a + 2/(a*(tan(1/2*x) + 1)) + 1/12*(21*tan(1/2*x)^7 + 24*tan(1/2*x)^6
 + 45*tan(1/2*x)^5 + 120*tan(1/2*x)^4 - 45*tan(1/2*x)^3 + 136*tan(1/2*x)^2
- 21*tan(1/2*x) + 40)/((tan(1/2*x)^2 + 1)^4*a)
```

**maple [B]** time = 0.43, size = 185, normalized size = 2.80

$$\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)} + \frac{7\left(\tan^7\left(\frac{x}{2}\right)\right)}{4a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4} + \frac{2\left(\tan^6\left(\frac{x}{2}\right)\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4} + \frac{15\left(\tan^5\left(\frac{x}{2}\right)\right)}{4a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4} + \frac{10\left(\tan^4\left(\frac{x}{2}\right)\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4} - \frac{15\left(\tan^3\left(\frac{x}{2}\right)\right)}{4a\left(\tan^2\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^4/(a+a*csc(x)),x)
```

```
[Out] 2/a/(tan(1/2*x)+1)+7/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)^7+2/a/(tan(1/2*x)^2+
1)^4*tan(1/2*x)^6+15/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)^5+10/a/(tan(1/2*x)^2
+1)^4*tan(1/2*x)^4-15/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)^3+34/3/a/(tan(1/2*x
)^2+1)^4*tan(1/2*x)^2-7/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)+10/3/a/(tan(1/2*x
)^2+1)^4+15/4/a*arctan(tan(1/2*x))
```

**maxima [B]** time = 0.43, size = 230, normalized size = 3.48

$$\frac{\frac{19\sin(x)}{\cos(x)+1}+\frac{211\sin(x)^2}{(\cos(x)+1)^2}+\frac{91\sin(x)^3}{(\cos(x)+1)^3}+\frac{219\sin(x)^4}{(\cos(x)+1)^4}+\frac{165\sin(x)^5}{(\cos(x)+1)^5}+\frac{165\sin(x)^6}{(\cos(x)+1)^6}+\frac{45\sin(x)^7}{(\cos(x)+1)^7}+\frac{45\sin(x)^8}{(\cos(x)+1)^8}+64}{12\left(a+\frac{a\sin(x)}{\cos(x)+1}+\frac{4a\sin(x)^2}{(\cos(x)+1)^2}+\frac{4a\sin(x)^3}{(\cos(x)+1)^3}+\frac{6a\sin(x)^4}{(\cos(x)+1)^4}+\frac{6a\sin(x)^5}{(\cos(x)+1)^5}+\frac{4a\sin(x)^6}{(\cos(x)+1)^6}+\frac{4a\sin(x)^7}{(\cos(x)+1)^7}+\frac{a\sin(x)^8}{(\cos(x)+1)^8}+\frac{a\sin(x}{(\cos(x)+}\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^4/(a+a*csc(x)),x, algorithm="maxima")
```

```
[Out] 1/12*(19*sin(x)/(cos(x) + 1) + 211*sin(x)^2/(cos(x) + 1)^2 + 91*sin(x)^3/(c
os(x) + 1)^3 + 219*sin(x)^4/(cos(x) + 1)^4 + 165*sin(x)^5/(cos(x) + 1)^5 +
165*sin(x)^6/(cos(x) + 1)^6 + 45*sin(x)^7/(cos(x) + 1)^7 + 45*sin(x)^8/(cos
(x) + 1)^8 + 64)/(a + a*sin(x)/(cos(x) + 1) + 4*a*sin(x)^2/(cos(x) + 1)^2 +
 4*a*sin(x)^3/(cos(x) + 1)^3 + 6*a*sin(x)^4/(cos(x) + 1)^4 + 6*a*sin(x)^5/(
cos(x) + 1)^5 + 4*a*sin(x)^6/(cos(x) + 1)^6 + 4*a*sin(x)^7/(cos(x) + 1)^7 +
 a*sin(x)^8/(cos(x) + 1)^8 + a*sin(x)^9/(cos(x) + 1)^9) + 15/4*arctan(sin(x
)/(cos(x) + 1))/a
```

**mupad [B]** time = 0.40, size = 93, normalized size = 1.41

$$\frac{15\,x}{8\,a}+\frac{\frac{15\tan\left(\frac{x}{2}\right)^8}{4}+\frac{15\tan\left(\frac{x}{2}\right)^7}{4}+\frac{55\tan\left(\frac{x}{2}\right)^6}{4}+\frac{55\tan\left(\frac{x}{2}\right)^5}{4}+\frac{73\tan\left(\frac{x}{2}\right)^4}{4}+\frac{91\tan\left(\frac{x}{2}\right)^3}{12}+\frac{211\tan\left(\frac{x}{2}\right)^2}{12}+\frac{19\tan\left(\frac{x}{2}\right)}{12}+\frac{16}{3}}{a\left(\tan\left(\frac{x}{2}\right)^2+1\right)^4\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^4/(a + a/sin(x)),x)
```

```
[Out] (15*x)/(8*a) + ((19*tan(x/2))/12 + (211*tan(x/2)^2)/12 + (91*tan(x/2)^3)/12
 + (73*tan(x/2)^4)/4 + (55*tan(x/2)^5)/4 + (55*tan(x/2)^6)/4 + (15*tan(x/2)
^7)/4 + (15*tan(x/2)^8)/4 + 16/3)/(a*(tan(x/2)^2 + 1)^4*(tan(x/2) + 1))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int\frac{\sin^4(x)}{\csc(x)+1}\,dx}{a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)**4/(a+a*csc(x)),x)
```

```
[Out] Integral(sin(x)**4/(csc(x) + 1), x)/a
```

## 3.11 $\int \frac{1}{(a+a\csc(c+dx))^2}\,dx$

**Optimal**. Leaf size=57

$$\frac{4\cot(c+dx)}{3a^2d(\csc(c+dx)+1)}+\frac{x}{a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

```
[Out] x/a^2+4/3*cot(d*x+c)/a^2/d/(1+csc(d*x+c))+1/3*cot(d*x+c)/d/(a+a*csc(d*x+c))
^2
```

**Rubi [A]** time = 0.07, antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.250, Rules used = {3777, 3919, 3794}

$$\frac{4\cot(c+dx)}{3a^2d(\csc(c+dx)+1)}+\frac{x}{a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[c + d*x])^(-2),x]
```

```
[Out] x/a^2 + (4*Cot[c + d*x])/(3*a^2*d*(1 + Csc[c + d*x])) + Cot[c + d*x]/(3*d*(
a + a*Csc[c + d*x])^2)
```

Rule 3777

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(Cot[c
+ d*x]*(a + b*Csc[c + d*x])^n)/(d*(2*n + 1)), x] + Dist[1/(a^2*(2*n + 1)),
Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Intege
rQ[2*n]
```

Rule 3794

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> -Simp[Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{1}{(a+a\csc(c+dx))^2}\,dx &= \frac{\cot(c+dx)}{3d(a+a\csc(c+dx))^2}-\frac{\int \frac{-3a+a\csc(c+dx)}{a+a\csc(c+dx)}\,dx}{3a^2}\\ &= \frac{x}{a^2}+\frac{\cot(c+dx)}{3d(a+a\csc(c+dx))^2}-\frac{4\int \frac{\csc(c+dx)}{a+a\csc(c+dx)}\,dx}{3a}\\ &= \frac{x}{a^2}+\frac{\cot(c+dx)}{3d(a+a\csc(c+dx))^2}+\frac{4\cot(c+dx)}{3d\left(a^2+a^2\csc(c+dx)\right)}\end{aligned}$$

**Mathematica** [A] time = 0.33, size = 108, normalized size = 1.89

$$\frac{3(3c+3dx-4)\cos\left(\frac{1}{2}(c+dx)\right)+(-3c-3dx+10)\cos\left(\frac{3}{2}(c+dx)\right)+6\sin\left(\frac{1}{2}(c+dx)\right)((c+dx)\cos(c+dx)+}{6a^2d\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[c + d*x])^(-2),x]
```

```
[Out] (3*(-4 + 3*c + 3*d*x)*Cos[(c + d*x)/2] + (10 - 3*c - 3*d*x)*Cos[(3*(c + d*x
))/2] + 6*(-3 + 2*c + 2*d*x + (c + d*x)*Cos[c + d*x])*Sin[(c + d*x)/2])/(6*
a^2*d*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2])^3)
```

**fricas** [B] time = 0.47, size = 124, normalized size = 2.18

$$\frac{(3\,dx-5)\cos\left(dx+c\right)^2-6\,dx-(3\,dx+4)\cos\left(dx+c\right)-(6\,dx+(3\,dx+5)\cos\left(dx+c\right)+1)\sin\left(dx+c\right)+1}{3\left(a^2d\cos\left(dx+c\right)^2-a^2d\cos\left(dx+c\right)-2\,a^2d-\left(a^2d\cos\left(dx+c\right)+2\,a^2d\right)\sin\left(dx+c\right)\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))^2,x, algorithm="fricas")
```

```
[Out] 1/3*((3*d*x - 5)*cos(d*x + c)^2 - 6*d*x - (3*d*x + 4)*cos(d*x + c) - (6*d*x
 + (3*d*x + 5)*cos(d*x + c) + 1)*sin(d*x + c) + 1)/(a^2*d*cos(d*x + c)^2 -
a^2*d*cos(d*x + c) - 2*a^2*d - (a^2*d*cos(d*x + c) + 2*a^2*d)*sin(d*x + c))
```

**giac** [A] time = 0.74, size = 60, normalized size = 1.05

$$\frac{\frac{3\,(dx+c)}{a^2}+\frac{2\left(3\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2+9\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+4\right)}{a^2\left(\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+1\right)^3}}{3\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))^2,x, algorithm="giac")
```

```
[Out] 1/3*(3*(d*x + c)/a^2 + 2*(3*tan(1/2*d*x + 1/2*c)^2 + 9*tan(1/2*d*x + 1/2*c)
 + 4)/(a^2*(tan(1/2*d*x + 1/2*c) + 1)^3))/d
```

**maple** [A] time = 0.71, size = 83, normalized size = 1.46

$$-\frac{4}{3d\,a^2\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^3}+\frac{2}{d\,a^2\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^2}+\frac{2}{d\,a^2\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{d\,a^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+a*csc(d*x+c))^2,x)
```

```
[Out] -4/3/d/a^2/(tan(1/2*d*x+1/2*c)+1)^3+2/d/a^2/(tan(1/2*d*x+1/2*c)+1)^2+2/d/a^
2/(tan(1/2*d*x+1/2*c)+1)+2/d/a^2*arctan(tan(1/2*d*x+1/2*c))
```

**maxima** [B] time = 0.43, size = 142, normalized size = 2.49

$$\frac{2\left(\frac{\frac{9\sin(dx+c)}{\cos(dx+c)+1}+\frac{3\sin(dx+c)^2}{(\cos(dx+c)+1)^2}+4}{a^2+\frac{3\,a^2\sin(dx+c)}{\cos(dx+c)+1}+\frac{3\,a^2\sin(dx+c)^2}{(\cos(dx+c)+1)^2}+\frac{a^2\sin(dx+c)^3}{(\cos(dx+c)+1)^3}}+\frac{3\,\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{a^2}\right)}{3\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))^2,x, algorithm="maxima")
```

```
[Out] 2/3*((9*sin(d*x + c)/(cos(d*x + c) + 1) + 3*sin(d*x + c)^2/(cos(d*x + c) +
1)^2 + 4)/(a^2 + 3*a^2*sin(d*x + c)/(cos(d*x + c) + 1) + 3*a^2*sin(d*x + c)
^2/(cos(d*x + c) + 1)^2 + a^2*sin(d*x + c)^3/(cos(d*x + c) + 1)^3) + 3*arct
an(sin(d*x + c)/(cos(d*x + c) + 1))/a^2)/d
```

**mupad [B]** time = 0.41, size = 52, normalized size = 0.91

$$\frac{x}{a^2}+\frac{2\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2+6\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+\frac{8}{3}}{a^2\,d\,{\left(\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+1\right)}^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + a/sin(c + d*x))^2,x)
```

```
[Out] x/a^2 + (6*tan(c/2 + (d*x)/2) + 2*tan(c/2 + (d*x)/2)^2 + 8/3)/(a^2*d*(tan(c
/2 + (d*x)/2) + 1)^3)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{1}{\csc^{2}{\left(c+dx\right)}+2\csc{\left(c+dx\right)}+1}\, dx}{a^{2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))**2,x)
```

```
[Out] Integral(1/(csc(c + d*x)**2 + 2*csc(c + d*x) + 1), x)/a**2
```

## 3.12 $\int \frac{1}{(a+a\csc(c+dx))^3}\,dx$

**Optimal**. Leaf size=88

$$\frac{22\cot(c+dx)}{15d\left(a^3\csc(c+dx)+a^3\right)}+\frac{x}{a^3}+\frac{7\cot(c+dx)}{15ad(a\csc(c+dx)+a)^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

```
[Out] x/a^3+1/5*cot(d*x+c)/d/(a+a*csc(d*x+c))^3+7/15*cot(d*x+c)/a/d/(a+a*csc(d*x+
c))^2+22/15*cot(d*x+c)/d/(a^3+a^3*csc(d*x+c))
```

**Rubi [A]** time = 0.11, antiderivative size = 88, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.333, Rules used = {3777, 3922, 3919, 3794}

$$\frac{22\cot(c+dx)}{15d\left(a^3\csc(c+dx)+a^3\right)}+\frac{x}{a^3}+\frac{7\cot(c+dx)}{15ad(a\csc(c+dx)+a)^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[c + d*x])^(-3),x]
```

```
[Out] x/a^3 + Cot[c + d*x]/(5*d*(a + a*Csc[c + d*x])^3) + (7*Cot[c + d*x])/(15*a*
d*(a + a*Csc[c + d*x])^2) + (22*Cot[c + d*x])/(15*d*(a^3 + a^3*Csc[c + d*x]
))
```

Rule 3777

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(Cot[c
+ d*x]*(a + b*Csc[c + d*x])^n)/(d*(2*n + 1)), x] + Dist[1/(a^2*(2*n + 1)),
Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Intege
rQ[2*n]
```

Rule 3794

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> -Simp[Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rule 3922

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(csc[(e_.) + (f_.)*(x_)]*(d
_.) + (c_)), x_Symbol] :> -Simp[((b*c - a*d)*Cot[e + f*x]*(a + b*Csc[e + f*
x])^m)/(b*f*(2*m + 1)), x] + Dist[1/(a^2*(2*m + 1)), Int[(a + b*Csc[e + f*x
])^(m + 1)*Simp[a*c*(2*m + 1) - (b*c - a*d)*(m + 1)*Csc[e + f*x], x], x], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] && EqQ
[a^2 - b^2, 0] && IntegerQ[2*m]
```

Rubi steps

$$\int \frac{1}{(a+a\csc(c+dx))^3}\,dx = \frac{\cot(c+dx)}{5d(a+a\csc(c+dx))^3} - \frac{\int \frac{-5a+2a\csc(c+dx)}{(a+a\csc(c+dx))^2}\,dx}{5a^2}$$

$$= \frac{\cot(c+dx)}{5d(a+a\csc(c+dx))^3} + \frac{7\cot(c+dx)}{15ad(a+a\csc(c+dx))^2} + \frac{\int \frac{15a^2-7a^2\csc(c+dx)}{a+a\csc(c+dx)}\,dx}{15a^4}$$

$$= \frac{x}{a^3} + \frac{\cot(c+dx)}{5d(a+a\csc(c+dx))^3} + \frac{7\cot(c+dx)}{15ad(a+a\csc(c+dx))^2} - \frac{22\int \frac{\csc(c+dx)}{a+a\csc(c+dx)}\,dx}{15a^2}$$

$$= \frac{x}{a^3} + \frac{\cot(c+dx)}{5d(a+a\csc(c+dx))^3} + \frac{7\cot(c+dx)}{15ad(a+a\csc(c+dx))^2} + \frac{22\cot(c+dx)}{15d\left(a^3+a^3\csc(c+dx)\right)}$$

**Mathematica [A]** time = 1.04, size = 123, normalized size = 1.40

$$\frac{\frac{2\sin\left(\frac{1}{2}(c+dx)\right)(-51\sin(c+dx)+16\cos(2(c+dx))-38)}{\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)^5} - \frac{13}{\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)^2} + \frac{3}{\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)^4} + 15c + 15dx}{15a^3d}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[c + d*x])^(-3),x]
```

```
[Out] (15*c + 15*d*x + 3/(Cos[(c + d*x)/2] + Sin[(c + d*x)/2])^4 - 13/(Cos[(c + d
*x)/2] + Sin[(c + d*x)/2])^2 + (2*Sin[(c + d*x)/2]*(-38 + 16*Cos[2*(c + d*x
)] - 51*Sin[c + d*x]))/(Cos[(c + d*x)/2] + Sin[(c + d*x)/2])^5)/(15*a^3*d)
```

**fricas [B]** time = 0.52, size = 181, normalized size = 2.06

$$\frac{(15\,dx+32)\cos(dx+c)^3 + (45\,dx-19)\cos(dx+c)^2 - 60\,dx - 6\,(5\,dx+9)\cos(dx+c) + \left((15\,dx-32)\cos(dx\right.}{15\left(a^3d\cos(dx+c)^3 + 3\,a^3d\cos(dx+c)^2 - 2\,a^3d\cos(dx+c) - 4\,a^3d + \left(a^3d\cos(dx+c)^2\right.\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))^3,x, algorithm="fricas")
```

```
[Out] 1/15*((15*d*x + 32)*cos(d*x + c)^3 + (45*d*x - 19)*cos(d*x + c)^2 - 60*d*x
- 6*(5*d*x + 9)*cos(d*x + c) + ((15*d*x - 32)*cos(d*x + c)^2 - 60*d*x - 3*(
10*d*x + 17)*cos(d*x + c) + 3)*sin(d*x + c) - 3)/(a^3*d*cos(d*x + c)^3 + 3*
a^3*d*cos(d*x + c)^2 - 2*a^3*d*cos(d*x + c) - 4*a^3*d + (a^3*d*cos(d*x + c)
^2 - 2*a^3*d*cos(d*x + c) - 4*a^3*d)*sin(d*x + c))
```

**giac [A]** time = 0.45, size = 86, normalized size = 0.98

$$\frac{\frac{15\,(dx+c)}{a^3} + \frac{2\left(15\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^4 + 75\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^3 + 145\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2 + 95\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right) + 22\right)}{a^3\left(\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+1\right)^5}}{15\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))^3,x, algorithm="giac")
```

```
[Out] 1/15*(15*(d*x + c)/a^3 + 2*(15*tan(1/2*d*x + 1/2*c)^4 + 75*tan(1/2*d*x + 1/
2*c)^3 + 145*tan(1/2*d*x + 1/2*c)^2 + 95*tan(1/2*d*x + 1/2*c) + 22)/(a^3*(t
an(1/2*d*x + 1/2*c) + 1)^5))/d
```

**maple [A]** time = 0.79, size = 125, normalized size = 1.42

$$-\frac{4}{d\,a^{3}\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^{4}}+\frac{8}{5d\,a^{3}\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^{5}}+\frac{4}{3d\,a^{3}\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^{3}}+\frac{2}{d\,a^{3}\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^{2}}+\frac{}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+a*csc(d*x+c))^3,x)
```

```
[Out] -4/d/a^3/(tan(1/2*d*x+1/2*c)+1)^4+8/5/d/a^3/(tan(1/2*d*x+1/2*c)+1)^5+4/3/d/
a^3/(tan(1/2*d*x+1/2*c)+1)^3+2/d/a^3/(tan(1/2*d*x+1/2*c)+1)^2+2/d/a^3/(tan(
1/2*d*x+1/2*c)+1)+2/d/a^3*arctan(tan(1/2*d*x+1/2*c))
```

**maxima [B]** time = 0.44, size = 228, normalized size = 2.59

$$\frac{2\left(\frac{\frac{95\sin(dx+c)}{\cos(dx+c)+1}+\frac{145\sin(dx+c)^{2}}{(\cos(dx+c)+1)^{2}}+\frac{75\sin(dx+c)^{3}}{(\cos(dx+c)+1)^{3}}+\frac{15\sin(dx+c)^{4}}{(\cos(dx+c)+1)^{4}}+22}{a^{3}+\frac{5a^{3}\sin(dx+c)}{\cos(dx+c)+1}+\frac{10a^{3}\sin(dx+c)^{2}}{(\cos(dx+c)+1)^{2}}+\frac{10a^{3}\sin(dx+c)^{3}}{(\cos(dx+c)+1)^{3}}+\frac{5a^{3}\sin(dx+c)^{4}}{(\cos(dx+c)+1)^{4}}+\frac{a^{3}\sin(dx+c)^{5}}{(\cos(dx+c)+1)^{5}}}+\frac{15\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{a^{3}}\right)}{15d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))^3,x, algorithm="maxima")
```

```
[Out] 2/15*((95*sin(d*x + c)/(cos(d*x + c) + 1) + 145*sin(d*x + c)^2/(cos(d*x + c
) + 1)^2 + 75*sin(d*x + c)^3/(cos(d*x + c) + 1)^3 + 15*sin(d*x + c)^4/(cos(
d*x + c) + 1)^4 + 22)/(a^3 + 5*a^3*sin(d*x + c)/(cos(d*x + c) + 1) + 10*a^3
*sin(d*x + c)^2/(cos(d*x + c) + 1)^2 + 10*a^3*sin(d*x + c)^3/(cos(d*x + c)
+ 1)^3 + 5*a^3*sin(d*x + c)^4/(cos(d*x + c) + 1)^4 + a^3*sin(d*x + c)^5/(co
s(d*x + c) + 1)^5) + 15*arctan(sin(d*x + c)/(cos(d*x + c) + 1))/a^3)/d
```

**mupad [B]** time = 2.18, size = 78, normalized size = 0.89

$$\frac{x}{a^{3}}+\frac{2\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^{4}+10\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^{3}+\frac{58\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^{2}}{3}+\frac{38\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{3}+\frac{44}{15}}{a^{3}\,d\left(\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+1\right)^{5}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + a/sin(c + d*x))^3,x)
```

```
[Out] x/a^3 + ((38*tan(c/2 + (d*x)/2))/3 + (58*tan(c/2 + (d*x)/2)^2)/3 + 10*tan(c
/2 + (d*x)/2)^3 + 2*tan(c/2 + (d*x)/2)^4 + 44/15)/(a^3*d*(tan(c/2 + (d*x)/2
) + 1)^5)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int\frac{1}{\csc^{3}(c+dx)+3\csc^{2}(c+dx)+3\csc(c+dx)+1}\,dx}{a^{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(d*x+c))**3,x)
```

```
[Out] Integral(1/(csc(c + d*x)**3 + 3*csc(c + d*x)**2 + 3*csc(c + d*x) + 1), x)/a
**3
```

## 3.13 $\int (a + a\csc(x))^{5/2}\,dx$

**Optimal**. Leaf size=65

$$-2a^{5/2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)-\frac{14a^3\cot(x)}{3\sqrt{a\csc(x)+a}}-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

```
[Out] -2*a^(5/2)*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))-14/3*a^3*cot(x)/(a+a*c
sc(x))^(1/2)-2/3*a^2*cot(x)*(a+a*csc(x))^(1/2)
```

**Rubi [A]** time = 0.09, antiderivative size = 65, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {3775, 3915, 3774, 203, 3792}

$$-\frac{14a^3\cot(x)}{3\sqrt{a\csc(x)+a}}-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}-2a^{5/2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[x])^(5/2),x]
```

```
[Out] -2*a^(5/2)*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]] - (14*a^3*Cot[x])/(3
*Sqrt[a + a*Csc[x]]) - (2*a^2*Cot[x]*Sqrt[a + a*Csc[x]])/3
```

Rule 203

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTan[(Rt[b, 2]*x)/Rt
[a, 2]])/(Rt[a, 2]*Rt[b, 2]), x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 3774

```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[(-2*b)/d,
 Subst[Int[1/(a + x^2), x], x, (b*Cot[c + d*x])/Sqrt[a + b*Csc[c + d*x]]],
x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3775

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(b^2*Co
t[c + d*x]*(a + b*Csc[c + d*x])^(n - 2))/(d*(n - 1)), x] + Dist[a/(n - 1),
Int[(a + b*Csc[c + d*x])^(n - 2)*(a*(n - 1) + b*(3*n - 4)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && GtQ[n, 1] && Integer
Q[2*n]
```

Rule 3792

```
Int[csc[(e_.) + (f_.)*(x_)]*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Simp[(-2*b*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]]), x] /; Free
Q[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3915

```
Int[Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)]*(csc[(e_.) + (f_.)*(x_)]*(d_
.) + (c_)), x_Symbol] :> Dist[c, Int[Sqrt[a + b*Csc[e + f*x]], x], x] + Dis
t[d, Int[Sqrt[a + b*Csc[e + f*x]]*Csc[e + f*x], x], x] /; FreeQ[{a, b, c, d
, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int (a+a\csc(x))^{5/2}\,dx = -\frac{2}{3}a^2\cot(x)\sqrt{a+a\csc(x)} + \frac{1}{3}(2a)\int \sqrt{a+a\csc(x)}\left(\frac{3a}{2}+\frac{7}{2}a\csc(x)\right)dx$$
$$= -\frac{2}{3}a^2\cot(x)\sqrt{a+a\csc(x)} + a^2\int \sqrt{a+a\csc(x)}\,dx + \frac{1}{3}\left(7a^2\right)\int \csc(x)\sqrt{a+a\csc(x)}$$
$$= -\frac{14a^3\cot(x)}{3\sqrt{a+a\csc(x)}} - \frac{2}{3}a^2\cot(x)\sqrt{a+a\csc(x)} - \left(2a^3\right)\text{Subst}\left(\int \frac{1}{a+x^2}\,dx, x, \frac{a\cot(}{\sqrt{a+a\,c}}\right.$$
$$= -2a^{5/2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right) - \frac{14a^3\cot(x)}{3\sqrt{a+a\csc(x)}} - \frac{2}{3}a^2\cot(x)\sqrt{a+a\csc(x)}$$

**Mathematica [A]** time = 1.90, size = 80, normalized size = 1.23

$$-\frac{2a^2\sqrt{a(\csc(x)+1)}\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(\sqrt{\csc(x)-1}\,(\csc(x)+8)+3\tan^{-1}\left(\sqrt{\csc(x)-1}\right)\right)}{3\sqrt{\csc(x)-1}\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[x])^(5/2),x]
```

```
[Out] (-2*a^2*Sqrt[a*(1 + Csc[x])]*(3*ArcTan[Sqrt[-1 + Csc[x]]] + Sqrt[-1 + Csc[x
]]*(8 + Csc[x]))*(Cos[x/2] - Sin[x/2]))/(3*Sqrt[-1 + Csc[x]]*(Cos[x/2] + Si
n[x/2]))
```

**fricas [B]** time = 0.56, size = 318, normalized size = 4.89

$$\left[\frac{3\left(a^2\cos(x)^2 - a^2 - \left(a^2\cos(x)+a^2\right)\sin(x)\right)\sqrt{-a}\,\log\left(\frac{2\,a\cos(x)^2-2\left(\cos(x)^2+(\cos(x)+1)\sin(x)-1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}+a\cos(x)-}{\cos(x)+\sin(x)+1}\right.}{3\left(\cos(x)^2-(\cos(x)+1)\right.}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(5/2),x, algorithm="fricas")
```

```
[Out] [1/3*(3*(a^2*cos(x)^2 - a^2 - (a^2*cos(x) + a^2)*sin(x))*sqrt(-a)*log((2*a*
cos(x)^2 - 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) +
 a)/sin(x)) + a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)
) + 2*(8*a^2*cos(x)^2 + a^2*cos(x) - 7*a^2 + (8*a^2*cos(x) + 7*a^2)*sin(x))
*sqrt((a*sin(x) + a)/sin(x)))/(cos(x)^2 - (cos(x) + 1)*sin(x) - 1), 2/3*(3*
(a^2*cos(x)^2 - a^2 - (a^2*cos(x) + a^2)*sin(x))*sqrt(a)*arctan(-sqrt(a)*sq
rt((a*sin(x) + a)/sin(x))*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a))
+ (8*a^2*cos(x)^2 + a^2*cos(x) - 7*a^2 + (8*a^2*cos(x) + 7*a^2)*sin(x))*sqr
t((a*sin(x) + a)/sin(x)))/(cos(x)^2 - (cos(x) + 1)*sin(x) - 1)]
```

**giac [B]** time = 1.01, size = 250, normalized size = 3.85

$$\left(a^2\sqrt{|a|} + a|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|} + 2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right) + \left(a^2\sqrt{|a|} + a|a|^{\frac{3}{2}}\right)\arctan\left(-\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|} - 2\sqrt{a\,\mathrm{ta}}\right.}{2\sqrt{|a|}}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(5/2),x, algorithm="giac")
```

```
[Out] (a^2*sqrt(abs(a)) + a*abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)
) + 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a))) + (a^2*sqrt(abs(a)) + a*abs(a)^(3/2
))*arctan(-1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(a
bs(a))) + 1/2*(a^2*sqrt(abs(a)) - a*abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2
)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a)) - 1/2*(a^2*sqrt(abs(a)) - a*abs
(a)^(3/2))*log(a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs
(a)) + 1/6*sqrt(2)*(sqrt(a*tan(1/2*x))*a^2*tan(1/2*x) + 15*sqrt(a*tan(1/2*x
))*a^2) - 1/6*sqrt(2)*(15*a^4*tan(1/2*x) + a^4)/(sqrt(a*tan(1/2*x))*a*tan(1
/2*x))
```

**maple [B]** time = 0.92, size = 535, normalized size = 8.23

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(x))^(5/2),x)
```

```
[Out] 1/6*(-3*sin(x)*cos(x)*(-(-1+cos(x))/sin(x))^(3/2)*ln(-(2^(1/2)*(-(-1+cos(x)
)/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2
)*sin(x)-sin(x)+cos(x)-1))-12*sin(x)*cos(x)*(-(-1+cos(x))/sin(x))^(3/2)*arc
tan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)+1)-12*sin(x)*cos(x)*(-(-1+cos(x))/s
in(x))^(3/2)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)-3*sin(x)*cos(x)*
(-(-1+cos(x))/sin(x))^(3/2)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)
-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)
+1))-3*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x)
)^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)
-sin(x)+cos(x)-1))-12*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)*arctan(2^(1/2)*(-(
-1+cos(x))/sin(x))^(1/2)+1)-12*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)*arctan(2^
(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)-3*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)*l
n(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-
(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))+16*sin(x)*cos(x)*2^(1/2)
-16*cos(x)^2*2^(1/2)-14*sin(x)*2^(1/2)+2*cos(x)*2^(1/2)+14*2^(1/2))*sin(x)*
(a*(1+sin(x))/sin(x))^(5/2)/(cos(x)^2*sin(x)-cos(x)^3+2*cos(x)*sin(x)+3*cos
(x)^2-4*sin(x)+2*cos(x)-4)*2^(1/2)
```

**maxima [B]** time = 0.47, size = 417, normalized size = 6.42

$$\frac{1}{22}\sqrt{2}\,a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{11}{2}}+\frac{5}{18}\sqrt{2}\,a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{9}{2}}+\frac{9}{14}\sqrt{2}\,a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{7}{2}}+\frac{1}{2}\sqrt{2}\,a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{5}{2}}-\frac{2}{3}\sqrt{2}\,a^{\frac{5}{2}}\left(\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(5/2),x, algorithm="maxima")
```

```
[Out] 1/22*sqrt(2)*a^(5/2)*(sin(x)/(cos(x) + 1))^(11/2) + 5/18*sqrt(2)*a^(5/2)*(s
in(x)/(cos(x) + 1))^(9/2) + 9/14*sqrt(2)*a^(5/2)*(sin(x)/(cos(x) + 1))^(7/2
) + 1/2*sqrt(2)*a^(5/2)*(sin(x)/(cos(x) + 1))^(5/2) - 2/3*sqrt(2)*a^(5/2)*(
sin(x)/(cos(x) + 1))^(3/2) + sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) +
 2*sqrt(sin(x)/(cos(x) + 1)))) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*s
qrt(sin(x)/(cos(x) + 1)))))*a^(5/2) - 2*sqrt(2)*a^(5/2)*sqrt(sin(x)/(cos(x)
 + 1)) - 1/1386*(693*sqrt(2)*a^(5/2)*sin(x)/(cos(x) + 1) + 1155*sqrt(2)*a^(
5/2)*sin(x)^2/(cos(x) + 1)^2 + 1386*sqrt(2)*a^(5/2)*sin(x)^3/(cos(x) + 1)^3
 + 990*sqrt(2)*a^(5/2)*sin(x)^4/(cos(x) + 1)^4 + 385*sqrt(2)*a^(5/2)*sin(x)
^5/(cos(x) + 1)^5 + 63*sqrt(2)*a^(5/2)*sin(x)^6/(cos(x) + 1)^6)/sqrt(sin(x)
/(cos(x) + 1)) - 1/42*(7*sqrt(2)*a^(5/2)*sin(x)/(cos(x) + 1) + 105*sqrt(2)*
a^(5/2)*sin(x)^2/(cos(x) + 1)^2 - 210*sqrt(2)*a^(5/2)*sin(x)^3/(cos(x) + 1)
^3 - 70*sqrt(2)*a^(5/2)*sin(x)^4/(cos(x) + 1)^4 - 21*sqrt(2)*a^(5/2)*sin(x)
```

```
^5/(cos(x) + 1)^5 - 3*sqrt(2)*a^(5/2)*sin(x)^6/(cos(x) + 1)^6)/(sin(x)/(cos
(x) + 1))^(5/2)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.02

$$\int \left(a+\frac{a}{\sin(x)}\right)^{5/2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(x))^(5/2),x)
```

```
[Out] int((a + a/sin(x))^(5/2), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int (a\csc(x)+a)^{\frac{5}{2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))**(5/2),x)
```

```
[Out] Integral((a*csc(x) + a)**(5/2), x)
```

## 3.14 $\int (a + a\csc(x))^{3/2}\, dx$

**Optimal**. Leaf size=44

$$-2a^{3/2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}$$

```
[Out] -2*a^(3/2)*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))-2*a^2*cot(x)/(a+a*csc(
x))^(1/2)
```

**Rubi [A]** time = 0.03, antiderivative size = 44, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.400, Rules used = {3775, 21, 3774, 203}

$$-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}-2a^{3/2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[x])^(3/2),x]
```

```
[Out] -2*a^(3/2)*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]] - (2*a^2*Cot[x])/Sqr
t[a + a*Csc[x]]
```

Rule 21

```
Int[(u_.)*((a_) + (b_.)*(v_))^(m_.)*((c_) + (d_.)*(v_))^(n_.), x_Symbol] :>
 Dist[(b/d)^m, Int[u*(c + d*v)^(m + n), x], x] /; FreeQ[{a, b, c, d, n}, x]
 && EqQ[b*c - a*d, 0] && IntegerQ[m] && ( !IntegerQ[n] || SimplerQ[c + d*x,
 a + b*x])
```

Rule 203

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTan[(Rt[b, 2]*x)/Rt
[a, 2]])/(Rt[a, 2]*Rt[b, 2]), x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 3774

```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[(-2*b)/d,
 Subst[Int[1/(a + x^2), x], x, (b*Cot[c + d*x])/Sqrt[a + b*Csc[c + d*x]]],
x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3775

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(b^2*Co
t[c + d*x]*(a + b*Csc[c + d*x])^(n - 2))/(d*(n - 1)), x] + Dist[a/(n - 1),
Int[(a + b*Csc[c + d*x])^(n - 2)*(a*(n - 1) + b*(3*n - 4)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && GtQ[n, 1] && Integer
Q[2*n]
```

Rubi steps

$$\int (a+a\csc(x))^{3/2}\,dx = -\frac{2a^2\cot(x)}{\sqrt{a+a\csc(x)}} + (2a)\int \frac{\frac{a}{2}+\frac{1}{2}a\csc(x)}{\sqrt{a+a\csc(x)}}\,dx$$
$$= -\frac{2a^2\cot(x)}{\sqrt{a+a\csc(x)}} + a\int \sqrt{a+a\csc(x)}\,dx$$
$$= -\frac{2a^2\cot(x)}{\sqrt{a+a\csc(x)}} - \left(2a^2\right)\text{Subst}\left(\int \frac{1}{a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)$$
$$= -2a^{3/2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right) - \frac{2a^2\cot(x)}{\sqrt{a+a\csc(x)}}$$

**Mathematica [A]** time = 0.09, size = 69, normalized size = 1.57

$$-\frac{2a\sqrt{a(\csc(x)+1)}\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(\sqrt{\csc(x)-1}+\tan^{-1}\left(\sqrt{\csc(x)-1}\right)\right)}{\sqrt{\csc(x)-1}\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[x])^(3/2),x]
```

```
[Out] (-2*a*(ArcTan[Sqrt[-1 + Csc[x]]] + Sqrt[-1 + Csc[x]])*Sqrt[a*(1 + Csc[x])]*
(Cos[x/2] - Sin[x/2]))/(Sqrt[-1 + Csc[x]]*(Cos[x/2] + Sin[x/2]))
```

**fricas [B]** time = 0.46, size = 212, normalized size = 4.82

$$\left[\frac{(a\cos(x)+a\sin(x)+a)\sqrt{-a}\,\log\left(\frac{2\,a\cos(x)^2-2\left(\cos(x)^2+(\cos(x)+1)\sin(x)-1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}+a\cos(x)-(2\,a\cos(x)+a)\sin(x)-a}{\cos(x)+\sin(x)+1}\right)}{\cos(x)+\sin(x)+1}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(3/2),x, algorithm="fricas")
```

```
[Out] [((a*cos(x) + a*sin(x) + a)*sqrt(-a)*log((2*a*cos(x)^2 - 2*(cos(x)^2 + (cos
(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x)) + a*cos(x) - (2*a
*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)) - 2*(a*cos(x) - a*sin(x) +
a)*sqrt((a*sin(x) + a)/sin(x)))/(cos(x) + sin(x) + 1), 2*((a*cos(x) + a*sin
(x) + a)*sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) - sin(
x) + 1)/(a*cos(x) + a*sin(x) + a)) - (a*cos(x) - a*sin(x) + a)*sqrt((a*sin(
x) + a)/sin(x)))/(cos(x) + sin(x) + 1)]
```

**giac [B]** time = 1.16, size = 195, normalized size = 4.43

$$\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\,a-\frac{\sqrt{2}\,a^2}{\sqrt{a\tan\left(\frac{1}{2}x\right)}}+\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}+2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)+\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)a$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(3/2),x, algorithm="giac")
```

```
[Out] sqrt(2)*sqrt(a*tan(1/2*x))*a - sqrt(2)*a^2/sqrt(a*tan(1/2*x)) + (a*sqrt(abs
(a)) + abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*ta
```

```
n(1/2*x)))/sqrt(abs(a))) + (a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/2*sqrt
(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a))) + 1/2*(a*sq
rt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*x))*sq
rt(abs(a)) + abs(a)) - 1/2*(a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x)
 - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))
```

**maple [B]** time = 0.63, size = 273, normalized size = 6.20

$$-\frac{\left(\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\ \ln\left(-\frac{\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\ \sin(x)+\sin(x)-\cos(x)+1}{\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\ \sin(x)-\sin(x)+\cos(x)-1}\right)\sin(x)+4\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\ \arctan\left(\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}+1\right)\sin\right.}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(x))^(3/2),x)
```

```
[Out] -1/2*((-(-1+cos(x))/sin(x))^(1/2)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*
sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+
cos(x)-1))*sin(x)+4*(-(-1+cos(x))/sin(x))^(1/2)*arctan(2^(1/2)*(-(-1+cos(x)
)/sin(x))^(1/2)+1)*sin(x)+4*(-(-1+cos(x))/sin(x))^(1/2)*arctan(2^(1/2)*(-(-
1+cos(x))/sin(x))^(1/2)-1)*sin(x)+(-(-1+cos(x))/sin(x))^(1/2)*ln(-(2^(1/2)*
(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/
sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))*sin(x)-2*sin(x)*2^(1/2)-2*cos(x)*2^(
1/2)+2*2^(1/2))*sin(x)*(a*(1+sin(x))/sin(x))^(3/2)/(cos(x)*sin(x)+cos(x)^2-
2*sin(x)+cos(x)-2)*2^(1/2)
```

**maxima [B]** time = 0.45, size = 200, normalized size = 4.55

$$\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)+\sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)\right)a^{\frac{3}{2}}-\frac{1}{5}\sqrt{2}\left(a^{\frac{3}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(3/2),x, algorithm="maxima")
```

```
[Out] sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x) + 1)))
) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1)))))*a
^(3/2) - 1/5*sqrt(2)*(a^(3/2)*(sin(x)/(cos(x) + 1))^(5/2) + 5*a^(3/2)*(sin(
x)/(cos(x) + 1))^(3/2) + 10*a^(3/2)*sqrt(sin(x)/(cos(x) + 1))) - 1/5*(5*sqr
t(2)*a^(3/2)*sin(x)/(cos(x) + 1) - 15*sqrt(2)*a^(3/2)*sin(x)^2/(cos(x) + 1)
^2 - 5*sqrt(2)*a^(3/2)*sin(x)^3/(cos(x) + 1)^3 - sqrt(2)*a^(3/2)*sin(x)^4/(
cos(x) + 1)^4)/(sin(x)/(cos(x) + 1))^(3/2)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.02

$$\int \left(a+\frac{a}{\sin(x)}\right)^{3/2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(x))^(3/2),x)
```

```
[Out] int((a + a/sin(x))^(3/2), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int (a\csc(x)+a)^{\frac{3}{2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))**(3/2),x)
```

```
[Out] Integral((a*csc(x) + a)**(3/2), x)
```

## 3.15 $\int \sqrt{a+a\csc(x)}\,dx$

**Optimal**. Leaf size=26

$$-2\sqrt{a}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)$$

```
[Out] -2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))*a^(1/2)
```

**Rubi [A]** time = 0.02, antiderivative size = 26, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {3774, 203}

$$-2\sqrt{a}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + a*Csc[x]],x]
```

```
[Out] -2*Sqrt[a]*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]]
```

Rule 203

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTan[(Rt[b, 2]*x)/Rt
[a, 2]])/(Rt[a, 2]*Rt[b, 2]), x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 3774

```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[(-2*b)/d,
 Subst[Int[1/(a + x^2), x], x, (b*Cot[c + d*x])/Sqrt[a + b*Csc[c + d*x]]],
x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\begin{aligned}\int \sqrt{a+a\csc(x)}\,dx &= -\left((2a)\,\text{Subst}\left(\int \frac{1}{a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)\right)\\ &= -2\sqrt{a}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)\end{aligned}$$

**Mathematica [A]** time = 0.05, size = 32, normalized size = 1.23

$$-\frac{2a\cot(x)\tan^{-1}\left(\sqrt{\csc(x)-1}\right)}{\sqrt{\csc(x)-1}\sqrt{a(\csc(x)+1)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + a*Csc[x]],x]
```

```
[Out] (-2*a*ArcTan[Sqrt[-1 + Csc[x]]]*Cot[x])/(Sqrt[-1 + Csc[x]]*Sqrt[a*(1 + Csc[
x])])
```

**fricas [B]** time = 0.55, size = 120, normalized size = 4.62

$$\left[\sqrt{-a}\log\left(\frac{2\,a\cos(x)^2 - 2\left(\cos(x)^2 + (\cos(x)+1)\sin(x) - 1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}} + a\cos(x) - (2\,a\cos(x)+a)\sin(x)}{\cos(x)+\sin(x)+1}\right.\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(1/2),x, algorithm="fricas")
```

```
[Out] [sqrt(-a)*log((2*a*cos(x)^2 - 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-
a)*sqrt((a*sin(x) + a)/sin(x)) + a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/(c
os(x) + sin(x) + 1)), 2*sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(x))
*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a))]
```

**giac [B]** time = 1.06, size = 353, normalized size = 13.58

$$\frac{1}{4}\sqrt{2}\left(\frac{2\sqrt{2}\left(a\sqrt{|a|}\,\mathrm{sgn}\left(\tan\left(\frac{1}{2}x\right)^{3}+\tan\left(\frac{1}{2}x\right)^{2}+\tan\left(\frac{1}{2}x\right)+1\right)+|a|^{\frac{3}{2}}\mathrm{sgn}\left(\tan\left(\frac{1}{2}x\right)^{3}+\tan\left(\frac{1}{2}x\right)^{2}+\tan\left(\frac{1}{2}x\right.\right.\right.}{a}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(1/2),x, algorithm="giac")
```

```
[Out] 1/4*sqrt(2)*(2*sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + ta
n(1/2*x) + 1) + abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x) +
 1))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*tan(1/2*x)))/sqrt(
abs(a)))/a + 2*sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + ta
n(1/2*x) + 1) + abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x) +
 1))*arctan(-1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt
(abs(a)))/a + sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan
(1/2*x) + 1) - abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x) +
1))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/a
- sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x) + 1)
 - abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x) + 1))*log(a*ta
n(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/a)*sgn(sin(x))
```

**maple [B]** time = 0.72, size = 199, normalized size = 7.65

$$\frac{\sqrt{2}\sqrt{\frac{a(1+\sin(x))}{\sin(x)}}\,\sin(x)\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\left(\ln\left(-\frac{\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\,\sin(x)+\sin(x)-\cos(x)+1}{\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\,\sin(x)-\sin(x)+\cos(x)-1}\right)+4\arctan\left(\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}+1\right.\right.}{2-2\cos(x)+2\sin(x)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(x))^(1/2),x)
```

```
[Out] 1/2*2^(1/2)*(a*(1+sin(x))/sin(x))^(1/2)*sin(x)*(-(-1+cos(x))/sin(x))^(1/2)*
(ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*
(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))+4*arctan(2^(1/2)*(-(-1
+cos(x))/sin(x))^(1/2)+1)+4*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)+l
n(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-
(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)))/(1-cos(x)+sin(x))
```

**maxima [B]** time = 0.44, size = 148, normalized size = 5.69

$$-\frac{2}{3}\sqrt{2}\sqrt{a}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{3}{2}}+\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)+\sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{}{c}}\right.\right.\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))^(1/2),x, algorithm="maxima")
```

```
[Out] -2/3*sqrt(2)*sqrt(a)*(sin(x)/(cos(x) + 1))^(3/2) + sqrt(2)*(sqrt(2)*arctan(
1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x) + 1)))) + sqrt(2)*arctan(-1/2*
sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1)))))*sqrt(a) - 2*sqrt(2)*sqrt(
a)*sqrt(sin(x)/(cos(x) + 1)) + 2/3*(3*sqrt(2)*sqrt(a)*sin(x)/(cos(x) + 1) +
 sqrt(2)*sqrt(a)*sin(x)^2/(cos(x) + 1)^2)/sqrt(sin(x)/(cos(x) + 1))
```

**mupad** [F] time = 0.00, size = -1, normalized size = -0.04

$$\int \sqrt{a + \frac{a}{\sin(x)}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(x))^(1/2),x)
```

```
[Out] int((a + a/sin(x))^(1/2), x)
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc{(x)} + a}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(x))**(1/2),x)
```

```
[Out] Integral(sqrt(a*csc(x) + a), x)
```

## 3.16 $\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx$

**Optimal**. Leaf size=62

$$\frac{\sqrt{2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}-\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}$$

```
[Out] -2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))/a^(1/2)+arctan(1/2*cot(x)*a^(1
/2)*2^(1/2)/(a+a*csc(x))^(1/2))*2^(1/2)/a^(1/2)
```

**Rubi [A]** time = 0.06, antiderivative size = 62, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.400, Rules used = {3776, 3774, 203, 3795}

$$\frac{\sqrt{2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}-\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}$$

Antiderivative was successfully verified.

```
[In] Int[1/Sqrt[a + a*Csc[x]],x]
```

```
[Out] (-2*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]])/Sqrt[a] + (Sqrt[2]*ArcTan[
(Sqrt[a]*Cot[x])/(Sqrt[2]*Sqrt[a + a*Csc[x]])])/Sqrt[a]
```

Rule 203

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTan[(Rt[b, 2]*x)/Rt
[a, 2]])/(Rt[a, 2]*Rt[b, 2]), x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 3774

```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[(-2*b)/d,
 Subst[Int[1/(a + x^2), x], x, (b*Cot[c + d*x])/Sqrt[a + b*Csc[c + d*x]]],
x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3776

```
Int[1/Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[1/a, In
t[Sqrt[a + b*Csc[c + d*x]], x], x] - Dist[b/a, Int[Csc[c + d*x]/Sqrt[a + b*
Csc[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3795

```
Int[csc[(e_.) + (f_.)*(x_)]/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Dist[-2/f, Subst[Int[1/(2*a + x^2), x], x, (b*Cot[e + f*x])/Sqrt[
a + b*Csc[e + f*x]]], x] /; FreeQ[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx = \frac{\int \sqrt{a+a\csc(x)}\,dx}{a} - \int \frac{\csc(x)}{\sqrt{a+a\csc(x)}}\,dx$$
$$= -\left(2\,\text{Subst}\left(\int \frac{1}{a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)\right) + 2\,\text{Subst}\left(\int \frac{1}{2a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)$$
$$= -\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{\sqrt{a}} + \frac{\sqrt{2}\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a+a\csc(x)}}\right)}{\sqrt{a}}$$

**Mathematica [A]** time = 0.14, size = 54, normalized size = 0.87

$$\frac{\cot(x)\left(\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\csc(x)-1}}{\sqrt{2}}\right) - 2\tan^{-1}\left(\sqrt{\csc(x)-1}\right)\right)}{\sqrt{\csc(x)-1}\sqrt{a(\csc(x)+1)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/Sqrt[a + a*Csc[x]],x]
[Out] ((-2*ArcTan[Sqrt[-1 + Csc[x]]] + Sqrt[2]*ArcTan[Sqrt[-1 + Csc[x]]/Sqrt[2]])
*Cot[x])/(Sqrt[-1 + Csc[x]]*Sqrt[a*(1 + Csc[x])])
```

**fricas [A]** time = 0.54, size = 219, normalized size = 3.53

$$\left[\frac{\sqrt{2}a\sqrt{-\frac{1}{a}}\log\left(\frac{\sqrt{2}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}\sqrt{-\frac{1}{a}}\sin(x)+\cos(x)}{\sin(x)+1}\right) - \sqrt{-a}\log\left(\frac{2a\cos(x)^2+2\left(\cos(x)^2+(\cos(x)+1)\sin(x)-1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}+a\cos(x)-}{\cos(x)+\sin(x)+1}\right.}{a}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(1/2),x, algorithm="fricas")
[Out] [(sqrt(2)*a*sqrt(-1/a)*log((sqrt(2)*sqrt((a*sin(x) + a)/sin(x))*sqrt(-1/a)*
sin(x) + cos(x))/(sin(x) + 1)) - sqrt(-a)*log((2*a*cos(x)^2 + 2*(cos(x)^2 +
 (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x)) + a*cos(x) -
 (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)))/a, -2*(sqrt(2)*sqrt(a
)*arctan(sqrt(2)*sqrt((a*sin(x) + a)/sin(x))*sin(x)/(sqrt(a)*(cos(x) + sin(
x) + 1))) - sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) - s
in(x) + 1)/(a*cos(x) + a*sin(x) + a)))/a]
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\sqrt{a\csc(x)+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(1/2),x, algorithm="giac")
[Out] integrate(1/sqrt(a*csc(x) + a), x)
```

**maple [B]** time = 0.74, size = 221, normalized size = 3.56

$$-\frac{\left(4\sqrt{2}\arctan\left(\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\right) - \ln\left(-\frac{\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\sin(x)+\sin(x)-\cos(x)+1}{\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}\sin(x)-\sin(x)+\cos(x)-1}\right) - 4\arctan\left(\sqrt{2}\sqrt{-\frac{-1+\cos(x)}{\sin(x)}}+1\right) - 4\,\text{ar}\right.}{4\sqrt{\frac{a(1+\sin(x))}{\sin(x)}}\sin(x)\sqrt{-}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+a*csc(x))^(1/2),x)
```

```
[Out] -1/4*(4*2^(1/2)*arctan((-(-1+cos(x))/sin(x))^(1/2))-ln(-(2^(1/2)*(-(-1+cos(
x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1
/2)*sin(x)-sin(x)+cos(x)-1))-4*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)+1
)-4*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)-ln(-(2^(1/2)*(-(-1+cos(x)
)/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2
)*sin(x)+sin(x)-cos(x)+1)))*(1-cos(x)+sin(x))/(a*(1+sin(x))/sin(x))^(1/2)/s
in(x)/(-(-1+cos(x))/sin(x))^(1/2)*2^(1/2)
```

**maxima [A]** time = 0.45, size = 83, normalized size = 1.34

$$\frac{\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)+\sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)\right)}{\sqrt{a}}-\frac{2\sqrt{2}\arctan\left(\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)}{\sqrt{a}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(1/2),x, algorithm="maxima")
```

```
[Out] sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x) + 1)))
) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1)))))/s
qrt(a) - 2*sqrt(2)*arctan(sqrt(sin(x)/(cos(x) + 1)))/sqrt(a)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.02

$$\int \frac{1}{\sqrt{a+\frac{a}{\sin(x)}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + a/sin(x))^(1/2),x)
```

```
[Out] int(1/(a + a/sin(x))^(1/2), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\sqrt{a\csc(x)+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))**(1/2),x)
```

```
[Out] Integral(1/sqrt(a*csc(x) + a), x)
```

## 3.17 $\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx$

**Optimal**. Leaf size=81

$$-\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{a^{3/2}}+\frac{5\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{2\sqrt{2}\,a^{3/2}}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

```
[Out] -2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))/a^(3/2)+1/2*cot(x)/(a+a*csc(x)
)^(3/2)+5/4*arctan(1/2*cot(x)*a^(1/2)*2^(1/2)/(a+a*csc(x))^(1/2))/a^(3/2)*2
^(1/2)
```

**Rubi [A]** time = 0.10, antiderivative size = 81, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {3777, 3920, 3774, 203, 3795}

$$-\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{a^{3/2}}+\frac{5\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{2\sqrt{2}\,a^{3/2}}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[x])^(-3/2),x]
```

```
[Out] (-2*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]])/a^(3/2) + (5*ArcTan[(Sqrt[
a]*Cot[x])/(Sqrt[2]*Sqrt[a + a*Csc[x]])])/(2*Sqrt[2]*a^(3/2)) + Cot[x]/(2*(
a + a*Csc[x])^(3/2))
```

Rule 203

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTan[(Rt[b, 2]*x)/Rt
[a, 2]])/(Rt[a, 2]*Rt[b, 2]), x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 3774

```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[(-2*b)/d,
 Subst[Int[1/(a + x^2), x], x, (b*Cot[c + d*x])/Sqrt[a + b*Csc[c + d*x]]],
x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3777

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(Cot[c
+ d*x]*(a + b*Csc[c + d*x])^n)/(d*(2*n + 1)), x] + Dist[1/(a^2*(2*n + 1)),
Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Intege
rQ[2*n]
```

Rule 3795

```
Int[csc[(e_.) + (f_.)*(x_)]/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Dist[-2/f, Subst[Int[1/(2*a + x^2), x], x, (b*Cot[e + f*x])/Sqrt[
a + b*Csc[e + f*x]]], x] /; FreeQ[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3920

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_
.) + (a_)], x_Symbol] :> Dist[c/a, Int[Sqrt[a + b*Csc[e + f*x]], x], x] - D
ist[(b*c - a*d)/a, Int[Csc[e + f*x]/Sqrt[a + b*Csc[e + f*x]], x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = \frac{\cot(x)}{2(a+a\csc(x))^{3/2}} - \frac{\int \frac{-2a+\frac{1}{2}a\csc(x)}{\sqrt{a+a\csc(x)}}\,dx}{2a^2}$$

$$= \frac{\cot(x)}{2(a+a\csc(x))^{3/2}} + \frac{\int \sqrt{a+a\csc(x)}\,dx}{a^2} - \frac{5\int \frac{\csc(x)}{\sqrt{a+a\csc(x)}}\,dx}{4a}$$

$$= \frac{\cot(x)}{2(a+a\csc(x))^{3/2}} - \frac{2\,\mathrm{Subst}\left(\int \frac{1}{a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{a} + \frac{5\,\mathrm{Subst}\left(\int \frac{1}{2a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{2a}$$

$$= -\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{a^{3/2}} + \frac{5\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a+a\csc(x)}}\right)}{2\sqrt{2}\,a^{3/2}} + \frac{\cot(x)}{2(a+a\csc(x))^{3/2}}$$

**Mathematica [A]** time = 0.43, size = 129, normalized size = 1.59

$$-\frac{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(-2\csc(x)+8\sqrt{\csc(x)-1}\,(\csc(x)+1)\tan^{-1}\left(\sqrt{\csc(x)-1}\right)-5\sqrt{2}\sqrt{\csc(x)-1}\,\csc(x)\left(\sin\ldots\right.\right.}{4a(\csc(x)-1)\sqrt{a(\csc(x)+1)}\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[x])^(-3/2),x]
```

```
[Out] -1/4*((Cos[x/2] - Sin[x/2])*(2 - 2*Csc[x] + 8*ArcTan[Sqrt[-1 + Csc[x]]]*Sqr
t[-1 + Csc[x]]*(1 + Csc[x]) - 5*Sqrt[2]*ArcTan[Sqrt[-1 + Csc[x]]/Sqrt[2]]*S
qrt[-1 + Csc[x]]*Csc[x]*(Cos[x/2] + Sin[x/2])^2))/(a*(-1 + Csc[x])*Sqrt[a*(
1 + Csc[x])]*(Cos[x/2] + Sin[x/2]))
```

**fricas [B]** time = 0.61, size = 427, normalized size = 5.27

$$\left[-\frac{5\sqrt{2}\left(\cos(x)^2-(\cos(x)+2)\sin(x)-\cos(x)-2\right)\sqrt{-a}\,\log\left(-\frac{\sqrt{2}\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}\sin(x)-a\cos(x)}{\sin(x)+1}\right)+4\left(\cos(x)^2-(\mathrm{c}\ldots\right.}{4\ldots}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(3/2),x, algorithm="fricas")
```

```
[Out] [-1/4*(5*sqrt(2)*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(-a)*log
(-(sqrt(2)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x))*sin(x) - a*cos(x))/(sin(x)
+ 1)) + 4*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(-a)*log((2*a*c
os(x)^2 + 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) +
a)/sin(x)) + a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1))
 + 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt((a*sin(x) + a)/sin(x)))/(a^2
*cos(x)^2 - a^2*cos(x) - 2*a^2 - (a^2*cos(x) + 2*a^2)*sin(x)), 1/2*(5*sqrt(
2)*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(a)*arctan(sqrt(2)*sqr
t(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) + 1)/(a*cos(x) + a*sin(x) + a)) +
4*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(a)*arctan(-sqrt(a)*sqr
t((a*sin(x) + a)/sin(x))*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a)) -
 (cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt((a*sin(x) + a)/sin(x)))/(a^2*cos
(x)^2 - a^2*cos(x) - 2*a^2 - (a^2*cos(x) + 2*a^2)*sin(x))]
```

**giac [B]** time = 1.93, size = 304, normalized size = 3.75

$$-\frac{1}{2}\left(\frac{5\sqrt{2}\arctan\left(\frac{\sqrt{a\tan\left(\frac{1}{2}x\right)}}{\sqrt{a}}\right)}{a^{\frac{3}{2}}\operatorname{sgn}\left(\tan\left(\frac{1}{2}x\right)+1\right)}-\frac{2\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}+2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)}{a^{3}\operatorname{sgn}\left(\tan\left(\frac{1}{2}x\right)+1\right)}-\frac{2\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(-\cdots\right.}{a^{3}\operatorname{sgn}\left(\tan\left(\cdots\right.\right.}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(3/2),x, algorithm="giac")
```

```
[Out] -1/2*(5*sqrt(2)*arctan(sqrt(a*tan(1/2*x))/sqrt(a))/(a^(3/2)*sgn(tan(1/2*x)
+ 1)) - 2*(a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(
abs(a)) + 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/(a^3*sgn(tan(1/2*x) + 1)) - 2
*(a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a))
- 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/(a^3*sgn(tan(1/2*x) + 1)) - (a*sqrt(a
bs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(a
bs(a)) + abs(a))/(a^3*sgn(tan(1/2*x) + 1)) + (a*sqrt(abs(a)) - abs(a)^(3/2)
)*log(a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/(a^3
*sgn(tan(1/2*x) + 1)) + sqrt(2)*(sqrt(a*tan(1/2*x))*a*tan(1/2*x) - sqrt(a*t
an(1/2*x))*a)/((a*tan(1/2*x) + a)^2*a*sgn(tan(1/2*x) + 1)))*sgn(tan(1/2*x))
```

**maple [B]** time = 0.81, size = 1141, normalized size = 14.09

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+a*csc(x))^(3/2),x)
```

```
[Out] -1/8*(-1+cos(x))*(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+4*ln(-(2^(1/2)
*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))
/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))+16*arctan(2^(1/2)*(-(-1+cos(x))/sin
(x))^(1/2)+1)+10*2^(1/2)*cos(x)*sin(x)*arctan((-(-1+cos(x))/sin(x))^(1/2))-
2^(1/2)*cos(x)*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)+2^(1/2)*cos(x)*sin(x)*(-(
-1+cos(x))/sin(x))^(1/2)+4*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-
sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+
1))+16*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)+2^(1/2)*cos(x)^2*(-(-1
+cos(x))/sin(x))^(3/2)-2*cos(x)*sin(x)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(
1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+si
n(x)-cos(x)+1))-2*cos(x)*sin(x)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*si
n(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+co
s(x)-1))-8*cos(x)*sin(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)+1)-8*co
s(x)*sin(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)+10*2^(1/2)*cos(x)
*arctan((-(-1+cos(x))/sin(x))^(1/2))-20*2^(1/2)*sin(x)*arctan((-(-1+cos(x))
/sin(x))^(1/2))+10*2^(1/2)*cos(x)^2*arctan((-(-1+cos(x))/sin(x))^(1/2))-2^(
1/2)*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)-2^(1/2)*cos(x)^2*(-(-1+cos(x))/sin(
x))^(1/2)-20*2^(1/2)*arctan((-(-1+cos(x))/sin(x))^(1/2))-2*cos(x)^2*ln(-(2^
(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+co
s(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))-2*cos(x)^2*ln(-(2^(1/2)*(-(-1+
cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x)
)^(1/2)*sin(x)-sin(x)+cos(x)-1))-8*cos(x)^2*arctan(2^(1/2)*(-(-1+cos(x))/si
n(x))^(1/2)+1)-8*cos(x)^2*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)-2*c
os(x)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(
1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))-2*cos(x)*ln(-(2^(
1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos
(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))-8*cos(x)*arctan(2^(1/2)*(-(-1+c
os(x))/sin(x))^(1/2)+1)-8*cos(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)
```

```
-1)+4*sin(x)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-
1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))+4*sin(x)*l
n(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-
(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))+16*sin(x)*arctan(2^(1/2)
*(-(-1+cos(x))/sin(x))^(1/2)+1)+16*sin(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(
x))^(1/2)-1)-2^(1/2)*(-(-1+cos(x))/sin(x))^(3/2)+2^(1/2)*(-(-1+cos(x))/sin(
x))^(1/2))/(a*(1+sin(x))/sin(x))^(3/2)/sin(x)^3/(-(-1+cos(x))/sin(x))^(3/2)
*2^(1/2)
```

**maxima** [B] time = 0.45, size = 150, normalized size = 1.85

$$-\frac{\sqrt{2}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{3}{2}}-\sqrt{2}\sqrt{\frac{\sin(x)}{\cos(x)+1}}}{2\left(a^{\frac{3}{2}}+\frac{2a^{\frac{3}{2}}\sin(x)}{\cos(x)+1}+\frac{a^{\frac{3}{2}}\sin(x)^{2}}{(\cos(x)+1)^{2}}\right)}+\frac{\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)+\sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\right.\right.\right.}{a^{\frac{3}{2}}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(3/2),x, algorithm="maxima")
```

```
[Out] -1/2*(sqrt(2)*(sin(x)/(cos(x) + 1))^(3/2) - sqrt(2)*sqrt(sin(x)/(cos(x) + 1
)))/(a^(3/2) + 2*a^(3/2)*sin(x)/(cos(x) + 1) + a^(3/2)*sin(x)^2/(cos(x) + 1
)^2) + sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x)
 + 1)))) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1
)))))/a^(3/2) - 5/2*sqrt(2)*arctan(sqrt(sin(x)/(cos(x) + 1)))/a^(3/2)
```

**mupad** [F] time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{1}{\left(a+\frac{a}{\sin(x)}\right)^{3/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + a/sin(x))^(3/2),x)
```

```
[Out] int(1/(a + a/sin(x))^(3/2), x)
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\left(a\csc(x)+a\right)^{\frac{3}{2}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))**(3/2),x)
```

```
[Out] Integral((a*csc(x) + a)**(-3/2), x)
```

## 3.18 $\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx$

**Optimal**. Leaf size=100

$$-\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{a^{5/2}}+\frac{43\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{16\sqrt{2}a^{5/2}}+\frac{11\cot(x)}{16a(a\csc(x)+a)^{3/2}}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

```
[Out] -2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))/a^(5/2)+1/4*cot(x)/(a+a*csc(x)
)^(5/2)+11/16*cot(x)/a/(a+a*csc(x))^(3/2)+43/32*arctan(1/2*cot(x)*a^(1/2)*2
^(1/2)/(a+a*csc(x))^(1/2))/a^(5/2)*2^(1/2)
```

**Rubi [A]** time = 0.15, antiderivative size = 100, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.600, Rules used = {3777, 3922, 3920, 3774, 203, 3795}

$$-\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{a^{5/2}}+\frac{43\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{16\sqrt{2}a^{5/2}}+\frac{11\cot(x)}{16a(a\csc(x)+a)^{3/2}}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[x])^(-5/2),x]
```

```
[Out] (-2*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]])/a^(5/2) + (43*ArcTan[(Sqrt
[a]*Cot[x])/(Sqrt[2]*Sqrt[a + a*Csc[x]])])/(16*Sqrt[2]*a^(5/2)) + Cot[x]/(4
*(a + a*Csc[x])^(5/2)) + (11*Cot[x])/(16*a*(a + a*Csc[x])^(3/2))
```

Rule 203

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTan[(Rt[b, 2]*x)/Rt
[a, 2]])/(Rt[a, 2]*Rt[b, 2]), x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 3774

```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Dist[(-2*b)/d,
 Subst[Int[1/(a + x^2), x], x, (b*Cot[c + d*x])/Sqrt[a + b*Csc[c + d*x]]],
x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3777

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(Cot[c
+ d*x]*(a + b*Csc[c + d*x])^n)/(d*(2*n + 1)), x] + Dist[1/(a^2*(2*n + 1)),
Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]), x]
, x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Intege
rQ[2*n]
```

Rule 3795

```
Int[csc[(e_.) + (f_.)*(x_)]/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Dist[-2/f, Subst[Int[1/(2*a + x^2), x], x, (b*Cot[e + f*x])/Sqrt[
a + b*Csc[e + f*x]]], x] /; FreeQ[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

Rule 3920

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_
.) + (a_)], x_Symbol] :> Dist[c/a, Int[Sqrt[a + b*Csc[e + f*x]], x], x] - D
ist[(b*c - a*d)/a, Int[Csc[e + f*x]/Sqrt[a + b*Csc[e + f*x]], x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

Rule 3922

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(csc[(e_.) + (f_.)*(x_)]*(d
_.) + (c_)), x_Symbol] :> -Simp[((b*c - a*d)*Cot[e + f*x]*(a + b*Csc[e + f*
x])^m)/(b*f*(2*m + 1)), x] + Dist[1/(a^2*(2*m + 1)), Int[(a + b*Csc[e + f*x
])^(m + 1)*Simp[a*c*(2*m + 1) - (b*c - a*d)*(m + 1)*Csc[e + f*x], x], x], x
] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] && EqQ
[a^2 - b^2, 0] && IntegerQ[2*m]
```

Rubi steps

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx = \frac{\cot(x)}{4(a+a\csc(x))^{5/2}} - \frac{\int \frac{-4a+\frac{3}{2}a\csc(x)}{(a+a\csc(x))^{3/2}}\,dx}{4a^2}$$

$$= \frac{\cot(x)}{4(a+a\csc(x))^{5/2}} + \frac{11\cot(x)}{16a(a+a\csc(x))^{3/2}} + \frac{\int \frac{8a^2-\frac{11}{4}a^2\csc(x)}{\sqrt{a+a\csc(x)}}\,dx}{8a^4}$$

$$= \frac{\cot(x)}{4(a+a\csc(x))^{5/2}} + \frac{11\cot(x)}{16a(a+a\csc(x))^{3/2}} + \frac{\int \sqrt{a+a\csc(x)}\,dx}{a^3} - \frac{43\int \frac{\csc(x)}{\sqrt{a+a\csc(x)}}\,dx}{32a^2}$$

$$= \frac{\cot(x)}{4(a+a\csc(x))^{5/2}} + \frac{11\cot(x)}{16a(a+a\csc(x))^{3/2}} - \frac{2\,\mathrm{Subst}\left(\int \frac{1}{a+x^2}\,dx, x, \frac{a\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{a^2} + \frac{43\,\mathrm{Su}}{}$$

$$= -\frac{2\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{a^{5/2}} + \frac{43\tan^{-1}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a+a\csc(x)}}\right)}{16\sqrt{2}\,a^{5/2}} + \frac{\cot(x)}{4(a+a\csc(x))^{5/2}} + \frac{11\cot(}{16a(a+a\,\mathrm{cs}}$$

**Mathematica [A]** time = 0.51, size = 139, normalized size = 1.39

$$\frac{\csc^2(x)\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)\left(8\sin(x)+15\cos(2x)-64\sqrt{\csc(x)-1}\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)^4\tan^{-1}\left(\sqrt{\csc(x)-1}\right)+\right.}{32(a(\csc(x)+1))^{5/2}\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[x])^(-5/2),x]
```

```
[Out] (Csc[x]^2*(Cos[x/2] + Sin[x/2])*(7 + 15*Cos[2*x] - 64*ArcTan[Sqrt[-1 + Csc[
x]]]*Sqrt[-1 + Csc[x]]*(Cos[x/2] + Sin[x/2])^4 + 43*Sqrt[2]*ArcTan[Sqrt[-1
+ Csc[x]]/Sqrt[2]]*Sqrt[-1 + Csc[x]]*(Cos[x/2] + Sin[x/2])^4 + 8*Sin[x]))/(
32*(a*(1 + Csc[x]))^(5/2)*(Cos[x/2] - Sin[x/2]))
```

**fricas [B]** time = 0.48, size = 546, normalized size = 5.46

$$\left[-\frac{43\sqrt{2}\left(\cos(x)^3+3\cos(x)^2+\left(\cos(x)^2-2\cos(x)-4\right)\sin(x)-2\cos(x)-4\right)\sqrt{-a}\log\left(-\frac{\sqrt{2}\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}\,\mathrm{sin}}{\sin(x)+1}\right.}{}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(5/2),x, algorithm="fricas")
```

```
[Out] [-1/32*(43*sqrt(2)*(cos(x)^3 + 3*cos(x)^2 + (cos(x)^2 - 2*cos(x) - 4)*sin(x
) - 2*cos(x) - 4)*sqrt(-a)*log(-(sqrt(2)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x
```

```
))*sin(x) - a*cos(x))/(sin(x) + 1)) + 32*(cos(x)^3 + 3*cos(x)^2 + (cos(x)^2
 - 2*cos(x) - 4)*sin(x) - 2*cos(x) - 4)*sqrt(-a)*log((2*a*cos(x)^2 + 2*(cos
(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x)) + a*c
os(x) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)) - 2*(15*cos(x)^
3 + 4*cos(x)^2 - (15*cos(x)^2 + 11*cos(x) - 4)*sin(x) - 15*cos(x) - 4)*sqrt
((a*sin(x) + a)/sin(x)))/(a^3*cos(x)^3 + 3*a^3*cos(x)^2 - 2*a^3*cos(x) - 4*
a^3 + (a^3*cos(x)^2 - 2*a^3*cos(x) - 4*a^3)*sin(x)), 1/16*(43*sqrt(2)*(cos(
x)^3 + 3*cos(x)^2 + (cos(x)^2 - 2*cos(x) - 4)*sin(x) - 2*cos(x) - 4)*sqrt(a
)*arctan(sqrt(2)*sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) + 1)/(a*cos(x)
 + a*sin(x) + a)) + 32*(cos(x)^3 + 3*cos(x)^2 + (cos(x)^2 - 2*cos(x) - 4)*s
in(x) - 2*cos(x) - 4)*sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(
cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a)) + (15*cos(x)^3 + 4*cos(x)^2
 - (15*cos(x)^2 + 11*cos(x) - 4)*sin(x) - 15*cos(x) - 4)*sqrt((a*sin(x) + a
)/sin(x)))/(a^3*cos(x)^3 + 3*a^3*cos(x)^2 - 2*a^3*cos(x) - 4*a^3 + (a^3*cos
(x)^2 - 2*a^3*cos(x) - 4*a^3)*sin(x))]
```

**giac** [B] time = 1.62, size = 348, normalized size = 3.48

$$-\frac{1}{16}\left(\frac{43\,\sqrt{2}\,\arctan\left(\frac{\sqrt{a\tan\left(\frac{1}{2}\,x\right)}}{\sqrt{a}}\right)}{a^{\frac{5}{2}}\mathrm{sgn}\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}-\frac{16\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\,\sqrt{|a|}+2\,\sqrt{a\tan\left(\frac{1}{2}\,x\right)}\right)}{2\,\sqrt{|a|}}\right)}{a^{4}\mathrm{sgn}\left(\tan\left(\frac{1}{2}\,x\right)+1\right)}-\frac{16\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\mathrm{arctan}}{a^{4}\mathrm{sgn}\left(t\right.}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(5/2),x, algorithm="giac")
```

```
[Out] -1/16*(43*sqrt(2)*arctan(sqrt(a*tan(1/2*x))/sqrt(a))/(a^(5/2)*sgn(tan(1/2*x
) + 1)) - 16*(a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sq
rt(abs(a)) + 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/(a^4*sgn(tan(1/2*x) + 1))
- 16*(a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/2*sqrt(2)*(sqrt(2)*sqrt(abs(
a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/(a^4*sgn(tan(1/2*x) + 1)) - 8*(a*
sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*x))*
sqrt(abs(a)) + abs(a))/(a^4*sgn(tan(1/2*x) + 1)) + 8*(a*sqrt(abs(a)) - abs(
a)^(3/2))*log(a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(
a))/(a^4*sgn(tan(1/2*x) + 1)) + sqrt(2)*(11*sqrt(a*tan(1/2*x))*a^3*tan(1/2*
x)^3 + 19*sqrt(a*tan(1/2*x))*a^3*tan(1/2*x)^2 - 19*sqrt(a*tan(1/2*x))*a^3*t
an(1/2*x) - 11*sqrt(a*tan(1/2*x))*a^3)/((a*tan(1/2*x) + a)^4*a^2*sgn(tan(1/
2*x) + 1)))*sgn(tan(1/2*x))
```

**maple** [B] time = 0.61, size = 1961, normalized size = 19.61

Expression too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+a*csc(x))^(5/2),x)
```

```
[Out] -1/128*(-1+cos(x))^2*(-11*2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-19*sin
(x)*cos(x)^2*(-(-1+cos(x))/sin(x))^(3/2)*2^(1/2)-11*sin(x)*cos(x)^2*(-(-1+c
os(x))/sin(x))^(1/2)*2^(1/2)+11*sin(x)*cos(x)^2*(-(-1+cos(x))/sin(x))^(7/2)
*2^(1/2)+19*sin(x)*cos(x)^2*(-(-1+cos(x))/sin(x))^(5/2)*2^(1/2)-172*sin(x)*
cos(x)^2*2^(1/2)*arctan((-(-1+cos(x))/sin(x))^(1/2))-11*cos(x)^3*(-(-1+cos(
x))/sin(x))^(7/2)*2^(1/2)-11*cos(x)^2*(-(-1+cos(x))/sin(x))^(7/2)*2^(1/2)+3
2*sin(x)*cos(x)^2*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-co
s(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))+128*s
in(x)*cos(x)^2*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)-128*ln(-(2^(1/
2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x
```

```
))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))-512*arctan(2^(1/2)*(-(-1+cos(x))/
sin(x))^(1/2)+1)-344*2^(1/2)*cos(x)*sin(x)*arctan((-(-1+cos(x))/sin(x))^(1/
2))-38*2^(1/2)*cos(x)*sin(x)*(-(-1+cos(x))/sin(x))^(3/2)-22*2^(1/2)*cos(x)*
sin(x)*(-(-1+cos(x))/sin(x))^(1/2)+38*sin(x)*cos(x)*(-(-1+cos(x))/sin(x))^(
5/2)*2^(1/2)+22*sin(x)*cos(x)*(-(-1+cos(x))/sin(x))^(7/2)*2^(1/2)-128*ln(-(
2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+
cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))-512*arctan(2^(1/2)*(-(-1+cos
(x))/sin(x))^(1/2)-1)+32*sin(x)*cos(x)^2*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))
^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+
sin(x)-cos(x)+1))+128*sin(x)*cos(x)^2*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^
(1/2)+1)+11*sin(x)*(-(-1+cos(x))/sin(x))^(7/2)*2^(1/2)-19*cos(x)^3*(-(-1+co
s(x))/sin(x))^(5/2)*2^(1/2)+11*cos(x)*(-(-1+cos(x))/sin(x))^(7/2)*2^(1/2)-1
9*cos(x)^2*(-(-1+cos(x))/sin(x))^(5/2)*2^(1/2)+19*sin(x)*(-(-1+cos(x))/sin(
x))^(5/2)*2^(1/2)+19*cos(x)^3*(-(-1+cos(x))/sin(x))^(3/2)*2^(1/2)+19*cos(x)
*(-(-1+cos(x))/sin(x))^(5/2)*2^(1/2)+11*cos(x)^3*(-(-1+cos(x))/sin(x))^(1/2
)*2^(1/2)-19*cos(x)*(-(-1+cos(x))/sin(x))^(3/2)*2^(1/2)+172*cos(x)^3*2^(1/2
)*arctan((-(-1+cos(x))/sin(x))^(1/2))-11*cos(x)*(-(-1+cos(x))/sin(x))^(1/2)
*2^(1/2)+19*2^(1/2)*cos(x)^2*(-(-1+cos(x))/sin(x))^(3/2)+64*cos(x)*sin(x)*l
n(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-
(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))+64*cos(x)*sin(x)*ln(-(2^
(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+co
s(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))+256*cos(x)*sin(x)*arctan(2^(1/
2)*(-(-1+cos(x))/sin(x))^(1/2)+1)+256*cos(x)*sin(x)*arctan(2^(1/2)*(-(-1+co
s(x))/sin(x))^(1/2)-1)-344*2^(1/2)*cos(x)*arctan((-(-1+cos(x))/sin(x))^(1/2
))+688*2^(1/2)*sin(x)*arctan((-(-1+cos(x))/sin(x))^(1/2))-516*2^(1/2)*cos(x
)^2*arctan((-(-1+cos(x))/sin(x))^(1/2))-19*2^(1/2)*sin(x)*(-(-1+cos(x))/sin
(x))^(3/2)+11*2^(1/2)*cos(x)^2*(-(-1+cos(x))/sin(x))^(1/2)+688*2^(1/2)*arct
an((-(-1+cos(x))/sin(x))^(1/2))+96*cos(x)^2*ln(-(2^(1/2)*(-(-1+cos(x))/sin(
x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(
x)+sin(x)-cos(x)+1))+96*cos(x)^2*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*s
in(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+c
os(x)-1))+384*cos(x)^2*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)+1)+384*co
s(x)^2*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)+64*cos(x)*ln(-(2^(1/2)
*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))
/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))+64*cos(x)*ln(-(2^(1/2)*(-(-1+cos(x)
)/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2
)*sin(x)-sin(x)+cos(x)-1))+256*cos(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^
(1/2)+1)+256*cos(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)-1)-128*sin(x
)*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1)/(2^(1/2)
*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1))-128*sin(x)*ln(-(2^(1/
2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x
))/sin(x))^(1/2)*sin(x)-sin(x)+cos(x)-1))-512*sin(x)*arctan(2^(1/2)*(-(-1+c
os(x))/sin(x))^(1/2)+1)-512*sin(x)*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/
2)-1)-19*2^(1/2)*(-(-1+cos(x))/sin(x))^(3/2)-32*cos(x)^3*ln(-(2^(1/2)*(-(-1
+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(x)+1)/(2^(1/2)*(-(-1+cos(x))/sin(x
))^(1/2)*sin(x)-sin(x)+cos(x)-1))-128*cos(x)^3*arctan(2^(1/2)*(-(-1+cos(x))
/sin(x))^(1/2)-1)-32*cos(x)^3*ln(-(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(
x)-sin(x)+cos(x)-1)/(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)*sin(x)+sin(x)-cos(
x)+1))-128*cos(x)^3*arctan(2^(1/2)*(-(-1+cos(x))/sin(x))^(1/2)+1)+19*(-(-1+
cos(x))/sin(x))^(5/2)*2^(1/2)+11*(-(-1+cos(x))/sin(x))^(7/2)*2^(1/2)-11*2^(
1/2)*(-(-1+cos(x))/sin(x))^(1/2))/(a*(1+sin(x))/sin(x))^(5/2)/sin(x)^5/(-(-
1+cos(x))/sin(x))^(5/2)*2^(1/2)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{(a\csc(x)+a)^{\frac{5}{2}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))^(5/2),x, algorithm="maxima")
```

```
[Out] integrate((a*csc(x) + a)^(-5/2), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{1}{\left(a+\frac{a}{\sin(x)}\right)^{5/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + a/sin(x))^(5/2),x)
```

```
[Out] int(1/(a + a/sin(x))^(5/2), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\left(a\csc\left(x\right)+a\right)^{\frac{5}{2}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+a*csc(x))**(5/2),x)
```

```
[Out] Integral((a*csc(x) + a)**(-5/2), x)
```

# 3.19 $\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx$

**Optimal**. Leaf size=37

$$-\frac{2\sqrt{a}\sinh^{-1}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a\csc(e+fx)+a}}\right)}{f}$$

```
[Out] -2*arcsinh(cot(f*x+e)*a^(1/2)/(a+a*csc(f*x+e))^(1/2))*a^(1/2)/f
```

**Rubi [A]** time = 0.06, antiderivative size = 37, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.080$, Rules used = $\{3801, 215\}$

$$-\frac{2\sqrt{a}\sinh^{-1}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a\csc(e+fx)+a}}\right)}{f}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[Csc[e + f*x]]*Sqrt[a + a*Csc[e + f*x]],x]
```

```
[Out] (-2*Sqrt[a]*ArcSinh[(Sqrt[a]*Cot[e + f*x])/Sqrt[a + a*Csc[e + f*x]]])/f
```

Rule 215

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSinh[(Rt[b, 2]*x)/Sqr
t[a]]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]
```

Rule 3801

```
Int[Sqrt[csc[(e_.) + (f_.)*(x_)]*(d_.)]*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
+ (a_)], x_Symbol] :> Dist[(-2*a*Sqrt[(a*d)/b])/(b*f), Subst[Int[1/Sqrt[1 +
 x^2/a], x], x, (b*Cot[e + f*x])/Sqrt[a + b*Csc[e + f*x]]], x] /; FreeQ[{a,
 b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && GtQ[(a*d)/b, 0]
```

Rubi steps

$$\begin{aligned}
\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx &= -\frac{2\,\mathrm{Subst}\left(\int \frac{1}{\sqrt{1+\frac{x^2}{a}}}\,dx, x, \frac{a\cot(e+fx)}{\sqrt{a+a\csc(e+fx)}}\right)}{f}\\
&= -\frac{2\sqrt{a}\sinh^{-1}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a+a\csc(e+fx)}}\right)}{f}
\end{aligned}$$

**Mathematica [B]** time = 0.40, size = 108, normalized size = 2.92

$$\frac{2\cot(e+fx)\sqrt{a(\csc(e+fx)+1)}\left(\log(\csc(e+fx)+1)-\log\left(\csc^{\frac{3}{2}}(e+fx)+\sqrt{\csc(e+fx)}+\sqrt{\cot^2(e+fx)}\right.\right.}{f\sqrt{\cot^2(e+fx)}\sqrt{\csc(e+fx)+1}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[Csc[e + f*x]]*Sqrt[a + a*Csc[e + f*x]],x]
```

```
[Out] (2*Cot[e + f*x]*Sqrt[a*(1 + Csc[e + f*x])]*(Log[1 + Csc[e + f*x]] - Log[Sqr
t[Csc[e + f*x]] + Csc[e + f*x]^(3/2) + Sqrt[Cot[e + f*x]^2]*Sqrt[1 + Csc[e
+ f*x]]]))/(f*Sqrt[Cot[e + f*x]^2]*Sqrt[1 + Csc[e + f*x]])
```

**fricas [B]** time = 0.80, size = 283, normalized size = 7.65

$$\left[\frac{\sqrt{a}\log\left(\frac{a\cos\left(fx+e\right)^{3}-7\,a\cos\left(fx+e\right)^{2}-9\,a\cos\left(fx+e\right)+\left(a\cos\left(fx+e\right)^{2}+8\,a\cos\left(fx+e\right)-a\right)\sin\left(fx+e\right)+\frac{4\left(\cos\left(fx+e\right)^{3}+3\cos\left(fx+e\right)^{2}-\left(\cos\left(fx+e\right)^{2}-2\cos\left(f\right.\right.\right.}{\sqrt{\sin}}}{\cos\left(fx+e\right)^{3}+\cos\left(fx+e\right)^{2}+\left(\cos\left(fx+e\right)^{2}-1\right)\sin\left(fx+e\right)-\cos\left(fx+e\right)-1}\right)}{2\,f}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x, algorithm="fricas")
```

```
[Out] [1/2*sqrt(a)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 9*a*cos(f*x + e)
+ (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) + 4*(cos(f*x + e)^
3 + 3*cos(f*x + e)^2 - (cos(f*x + e)^2 - 2*cos(f*x + e) - 3)*sin(f*x + e) -
 cos(f*x + e) - 3)*sqrt(a)*sqrt((a*sin(f*x + e) + a)/sin(f*x + e))/sqrt(sin
(f*x + e)) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin
(f*x + e) - cos(f*x + e) - 1))/f, sqrt(-a)*arctan(1/2*(cos(f*x + e)^2 + 2*s
in(f*x + e) - 1)*sqrt(-a)*sqrt((a*sin(f*x + e) + a)/sin(f*x + e))/(a*cos(f*
x + e)*sqrt(sin(f*x + e))))/f]
```

**giac [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: NotImplementedError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x, algorithm="giac")
```

```
[Out] Exception raised: NotImplementedError >> Unable to parse Giac output: Unabl
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qrt(a^3+a*(a*tan(1/2*(f*x+exp(1))))^2)-sqrt(a)*a*tan(1/2*(f*x+exp(1))))/sqr
t(-a)/a)/sqrt(-a)/a-1/2*sqrt(a)*ln(abs(sqrt(a^3+a*(a*tan(1/2*(f*x+exp(1))))
^2)-sqrt(a)*a*tan(1/2*(f*x+exp(1)))))/a)*sign(sin(f*x+exp(1)))/a/f
```

**maple [B]** time = 2.35, size = 113, normalized size = 3.05

$$\frac{\sqrt{2}\,\sqrt{\frac{1}{\sin\left(fx+e\right)}}\,\left(-1+\cos\left(fx+e\right)\right)\sqrt{\frac{a\left(1+\sin\left(fx+e\right)\right)}{\sin\left(fx+e\right)}}\,\left(\operatorname{arcsinh}\left(\frac{-1+\cos\left(fx+e\right)}{\sin\left(fx+e\right)}\right)+\operatorname{arctanh}\left(\frac{\sqrt{2}}{2\sqrt{\frac{1}{1+\cos\left(fx+e\right)}}}\right)\right)}{f\left(1-\cos\left(fx+e\right)+\sin\left(fx+e\right)\right)\sqrt{\frac{1}{1+\cos\left(fx+e\right)}}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x)
```

```
[Out] 1/f*2^(1/2)*(1/sin(f*x+e))^(1/2)*(-1+cos(f*x+e))*(a*(1+sin(f*x+e))/sin(f*x+
e))^(1/2)*(arcsinh((-1+cos(f*x+e))/sin(f*x+e))+arctanh(1/2*2^(1/2)/(1/(1+co
s(f*x+e)))^(1/2)))/(1-cos(f*x+e)+sin(f*x+e))/(1/(1+cos(f*x+e)))^(1/2)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a\csc\left(fx+e\right)+a}\,\sqrt{\csc\left(fx+e\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(a*csc(f*x + e) + a)*sqrt(csc(f*x + e)), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.03

$$\int \sqrt{a+\frac{a}{\sin\left(e+f\,x\right)}}\,\sqrt{\frac{1}{\sin\left(e+f\,x\right)}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(e + f*x))^(1/2)*(1/sin(e + f*x))^(1/2),x)
```

```
[Out] int((a + a/sin(e + f*x))^(1/2)*(1/sin(e + f*x))^(1/2), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a\left(\csc\left(e+fx\right)+1\right)}\,\sqrt{\csc\left(e+fx\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)**(1/2)*(a+a*csc(f*x+e))**(1/2),x)
```

```
[Out] Integral(sqrt(a*(csc(e + f*x) + 1))*sqrt(csc(e + f*x)), x)
```

## 3.20 $\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx$

**Optimal**. Leaf size=38

$$-\frac{2\sqrt{a}\sinh^{-1}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}$$

```
[Out] -2*arcsinh(cot(f*x+e)*a^(1/2)/(a-a*csc(f*x+e))^(1/2))*a^(1/2)/f
```

**Rubi [A]** time = 0.07, antiderivative size = 38, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.071$, Rules used = {3801, 215}

$$-\frac{2\sqrt{a}\sinh^{-1}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[-Csc[e + f*x]]*Sqrt[a - a*Csc[e + f*x]],x]
```

```
[Out] (-2*Sqrt[a]*ArcSinh[(Sqrt[a]*Cot[e + f*x])/Sqrt[a - a*Csc[e + f*x]]])/f
```

Rule 215

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSinh[(Rt[b, 2]*x)/Sqr
t[a]]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]
```

Rule 3801

```
Int[Sqrt[csc[(e_.) + (f_.)*(x_)]*(d_.)]*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
+ (a_)], x_Symbol] :> Dist[(-2*a*Sqrt[(a*d)/b])/(b*f), Subst[Int[1/Sqrt[1 +
 x^2/a], x], x, (b*Cot[e + f*x])/Sqrt[a + b*Csc[e + f*x]]], x] /; FreeQ[{a,
 b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && GtQ[(a*d)/b, 0]
```

Rubi steps

$$\begin{aligned}\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx &= \frac{2\,\text{Subst}\left(\int \frac{1}{\sqrt{1+\frac{x^2}{a}}}\,dx, x, -\frac{a\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}\\ &= -\frac{2\sqrt{a}\sinh^{-1}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}\end{aligned}$$

**Mathematica [B]** time = 0.79, size = 101, normalized size = 2.66

$$\frac{2\tan\left(\frac{1}{2}(e+fx)\right)\sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\left(\tanh^{-1}\left(\sqrt{\sec^2\left(\frac{1}{2}(e+fx)\right)}\right)+\sinh^{-1}\left(\tan\left(\frac{1}{2}(e+fx)\right)\right)\right)}{f\left(\tan\left(\frac{1}{2}(e+fx)\right)-1\right)\sqrt{\sec^2\left(\frac{1}{2}(e+fx)\right)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[-Csc[e + f*x]]*Sqrt[a - a*Csc[e + f*x]],x]
```

```
[Out] (2*(ArcSinh[Tan[(e + f*x)/2]] + ArcTanh[Sqrt[Sec[(e + f*x)/2]^2]])*Sqrt[-Cs
c[e + f*x]]*Sqrt[a - a*Csc[e + f*x]]*Tan[(e + f*x)/2])/(f*Sqrt[Sec[(e + f*x
)/2]^2]*(-1 + Tan[(e + f*x)/2]))
```

**fricas [B]** time = 1.08, size = 296, normalized size = 7.79

$$\left[\frac{\sqrt{a}\log\left(\frac{a\cos\left(fx+e\right)^{3}-7\,a\cos\left(fx+e\right)^{2}-4\left(\cos\left(fx+e\right)^{3}+3\,\cos\left(fx+e\right)^{2}+\left(\cos\left(fx+e\right)^{2}-2\,\cos\left(fx+e\right)-3\right)\sin\left(fx+e\right)-\cos\left(fx+e\right)-3\right)\sqrt{a}\sqrt{\frac{a\sin(fx\cdots}{\sin(fx\cdots}}}{\cos\left(fx+e\right)^{3}+\cos\left(fx+e\right)^{2}-\left(\cos\left(fx+e\right)^{2}-1\right)\sin\left(fx+e\right)-\cos(\cdots}\right)}{2\,f}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x, algorithm="fricas")
```

```
[Out] [1/2*sqrt(a)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^3
 + 3*cos(f*x + e)^2 + (cos(f*x + e)^2 - 2*cos(f*x + e) - 3)*sin(f*x + e) -
cos(f*x + e) - 3)*sqrt(a)*sqrt((a*sin(f*x + e) - a)/sin(f*x + e))*sqrt(-1/s
in(f*x + e)) - 9*a*cos(f*x + e) - (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)
*sin(f*x + e) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 - (cos(f*x + e)^2 - 1)*
sin(f*x + e) - cos(f*x + e) - 1))/f, sqrt(-a)*arctan(-1/2*(cos(f*x + e)^2 -
 2*sin(f*x + e) - 1)*sqrt(-a)*sqrt((a*sin(f*x + e) - a)/sin(f*x + e))*sqrt(
-1/sin(f*x + e))/(a*cos(f*x + e)))/f]
```

**giac [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: NotImplementedError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x, algorithm="giac")
```

```
[Out] Exception raised: NotImplementedError >> Unable to parse Giac output: Unabl
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n(abs(sqrt(a^3+a*(-a*tan(1/2*(f*x+exp(1))))^2)+sqrt(a)*a*tan(1/2*(f*x+exp(1
)))))-1/4*a^2*atan((sqrt(a^3+a*(-a*tan(1/2*(f*x+exp(1))))^2)+sqrt(a)*a*tan(
1/2*(f*x+exp(1))))/sqrt(-a)/a)/sqrt(-a)/a)/f
```

**maple [B]** time = 2.11, size = 117, normalized size = 3.08

$$-\frac{2\sqrt{-\frac{1}{\sin\left(fx+e\right)}}\left(-1+\cos\left(fx+e\right)\right)\sqrt{\frac{a\left(\sin\left(fx+e\right)-1\right)}{\sin\left(fx+e\right)}}\left(\arctan\left(\frac{1}{\sqrt{-\frac{2}{1+\cos\left(fx+e\right)}}}\right)-\arctan\left(\frac{\sin\left(fx+e\right)\sqrt{-\frac{2}{1+\cos\left(fx+e\right)}}}{2}\right)\right)}{f\left(-1+\cos\left(fx+e\right)+\sin\left(fx+e\right)\right)\sqrt{-\frac{2}{1+\cos\left(fx+e\right)}}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x)
```

```
[Out] -2/f*(-1/sin(f*x+e))^(1/2)*(-1+cos(f*x+e))*(a*(sin(f*x+e)-1)/sin(f*x+e))^(1
/2)*(arctan(1/(-2/(1+cos(f*x+e)))^(1/2))-arctan(1/2*sin(f*x+e)*(-2/(1+cos(f
*x+e)))^(1/2)))/(-1+cos(f*x+e)+sin(f*x+e))/(-2/(1+cos(f*x+e)))^(1/2)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{-a\csc\left(fx+e\right)+a}\sqrt{-\csc\left(fx+e\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(-a*csc(f*x + e) + a)*sqrt(-csc(f*x + e)), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.03

$$\int \sqrt{a-\frac{a}{\sin\left(e+fx\right)}}\sqrt{-\frac{1}{\sin\left(e+fx\right)}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a - a/sin(e + f*x))^(1/2)*(-1/sin(e + f*x))^(1/2),x)
```

```
[Out] int((a - a/sin(e + f*x))^(1/2)*(-1/sin(e + f*x))^(1/2), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{-\csc\left(e+fx\right)}\sqrt{-a\left(\csc\left(e+fx\right)-1\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-csc(f*x+e))**(1/2)*(a-a*csc(f*x+e))**(1/2),x)
```

```
[Out] Integral(sqrt(-csc(e + f*x))*sqrt(-a*(csc(e + f*x) - 1)), x)
```

## 3.21 $\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

**Optimal**. Leaf size=254

$$-\frac{4\,3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\middle|-7-\right.}{5d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

```
[Out] -6/5*a*cos(d*x+c)*csc(d*x+c)^(4/3)/d/(a+a*csc(d*x+c))^(1/2)-4/5*3^(3/4)*a^2
*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-
csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)+1/2*2^(1/2))*((1+csc(
d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a
*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/
3)+3^(1/2))^2)^(1/2)
```

**Rubi** [A] time = 0.28, antiderivative size = 254, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.160, Rules used = {3806, 50, 63, 218}

$$-\frac{4\,3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\middle|-7-\right.}{5d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[c + d*x]^(4/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (-6*a*Cos[c + d*x]*Csc[c + d*x]^(4/3))/(5*d*Sqrt[a + a*Csc[c + d*x]]) - (4*
3^(3/4)*Sqrt[2 + Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1
 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/
3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - C
sc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(5*d*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1
 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a + a*Csc[c +
 d*x]])
```

Rule 50

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
((a + b*x)^(m + 1)*(c + d*x)^n)/(b*(m + n + 1)), x] + Dist[(n*(b*c - a*d))/
(b*(m + n + 1)), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
```

```
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{\sqrt[3]{x}}{\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{6a\cos(c+dx)\csc^{\frac{4}{3}}(c+dx)}{5d\sqrt{a+a\csc(c+dx)}} + \frac{\left(2a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{x^{2/3}\sqrt{a-ax}}\,dx, \ldots\right)}{5d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(\ldots}}$$

$$= -\frac{6a\cos(c+dx)\csc^{\frac{4}{3}}(c+dx)}{5d\sqrt{a+a\csc(c+dx)}} + \frac{\left(6a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{\sqrt{a-ax^3}}\,dx, x, \ldots\right)}{5d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(\ldots}}$$

$$= -\frac{6a\cos(c+dx)\csc^{\frac{4}{3}}(c+dx)}{5d\sqrt{a+a\csc(c+dx)}} - \frac{4\,3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc\ldots}\right.}{5d\sqrt{\frac{1-\sqrt[3]{\csc\ldots}}{\left(1+\sqrt{3}-\sqrt[3]{\ldots}\right.}}}$$

**Mathematica [C]** time = 0.38, size = 102, normalized size = 0.40

$$-\frac{2\sqrt{a(\csc(c+dx)+1)}\left(2\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{3}{2};1-\csc(c+dx)\right)+3\sqrt[3]{\csc(c+dx)}\right)\left(\cos\left(\frac{1}{2}(c+dx)\right)-\sin\left(\frac{1}{2}(c+dx)\right)\right)}{5d\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^(4/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (-2*Sqrt[a*(1 + Csc[c + d*x])]*(3*Csc[c + d*x]^(1/3) + 2*Hypergeometric2F1[
1/2, 2/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2]))/(5
*d*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

**fricas [F]** time = 0.95, size = 0, normalized size = 0.00

$$\text{integral}\left(\sqrt{a\csc(dx+c)+a}\,\csc(dx+c)^{\frac{4}{3}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

```
[Out] integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(4/3), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a\csc(dx+c)+a}\,\csc(dx+c)^{\frac{4}{3}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(4/3), x)
```

**maple [F]** time = 2.64, size = 0, normalized size = 0.00

$$\int \left(\csc^{\frac{4}{3}}\left(dx+c\right)\right)\sqrt{a+a\csc\left(dx+c\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x)
[Out] int(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a\csc\left(dx+c\right)+a}\,\csc\left(dx+c\right)^{\frac{4}{3}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(4/3), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.00

$$\int \sqrt{a+\frac{a}{\sin\left(c+d\,x\right)}}\,{\left(\frac{1}{\sin\left(c+d\,x\right)}\right)}^{4/3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(4/3),x)
[Out] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(4/3), x)
```

**sympy [F(-1)]** time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**(4/3)*(a+a*csc(d*x+c))**(1/2),x)
[Out] Timed out
```

## 3.22 $\int \sqrt[3]{\csc(c+dx)}\, \sqrt{a+a\csc(c+dx)}\, dx$

**Optimal**. Leaf size=213

$$-\frac{2\, 3^{3/4}\sqrt{2+\sqrt{3}}\, a^2 \cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\, F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

```
[Out] -2*3^(3/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3
)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)+1/2*2^(
1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2
)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-
csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

**Rubi [A]** time = 0.13, antiderivative size = 213, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.120$, Rules used = {3806, 63, 218}

$$-\frac{2\, 3^{3/4}\sqrt{2+\sqrt{3}}\, a^2 \cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\, F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[c + d*x]^(1/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (-2*3^(3/4)*Sqrt[2 + Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3))*Sqr
t[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]
^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3]
 - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(d*Sqrt[(1 - Csc[c + d*x]^(1/3))/
(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a + a*Csc[c
 + d*x]])
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
```

```
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \sqrt[3]{\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}\,dx = \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{x^{2/3}\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}}$$

$$= \frac{\left(3a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{\sqrt{a-ax^3}}\,dx, x, \sqrt[3]{\csc(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{2\,3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\text{cs}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c}\right.}}}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\,(a-a\csc(c+dx}$$

**Mathematica [C]** time = 0.23, size = 46, normalized size = 0.22

$$-\frac{2a\cot(c+dx)\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{3}{2};1-\csc(c+dx)\right)}{d\sqrt{a(\csc(c+dx)+1)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^(1/3)*Sqrt[a + a*Csc[c + d*x]],x]
[Out] (-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 2/3, 3/2, 1 - Csc[c + d*x]])/(d*S
qrt[a*(1 + Csc[c + d*x])])
```

**fricas [F]** time = 0.91, size = 0, normalized size = 0.00

$$\text{integral}\left(\sqrt{a\csc\left(dx+c\right)+a}\,\csc\left(dx+c\right)^{\frac{1}{3}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
[Out] integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(1/3), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a\csc\left(dx+c\right)+a}\,\csc\left(dx+c\right)^{\frac{1}{3}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(1/3), x)
```

**maple [F]** time = 4.17, size = 0, normalized size = 0.00

$$\int \left(\csc^{\frac{1}{3}}\left(dx+c\right)\right)\sqrt{a+a\csc\left(dx+c\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x)
```

```
[Out] int(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x)
```

**maxima** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc (dx+c)+a} \csc (dx+c)^{\frac{1}{3}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(1/3), x)
```

**mupad** [F] time = 0.00, size = -1, normalized size = -0.00

$$\int \sqrt{a+\frac{a}{\sin (c+d\,x)}}\, {\left(\frac{1}{\sin (c+d\,x)}\right)}^{1/3} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(1/3),x)
[Out] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(1/3), x)
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a (\csc (c+dx)+1)} \sqrt[3]{\csc (c+dx)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**(1/3)*(a+a*csc(d*x+c))**(1/2),x)
[Out] Integral(sqrt(a*(csc(c + d*x) + 1))*csc(c + d*x)**(1/3), x)
```

## 3.23 $\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx$

**Optimal**. Leaf size=254

$$-\frac{3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\middle|\,-7-}{2d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

```
[Out] -3/2*a*cos(d*x+c)*csc(d*x+c)^(1/3)/d/(a+a*csc(d*x+c))^(1/2)-1/2*3^(3/4)*a^2
*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-
csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)+1/2*2^(1/2))*((1+csc(
d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a
*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/
3)+3^(1/2))^2)^(1/2)
```

**Rubi [A]** time = 0.14, antiderivative size = 254, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.160, Rules used = {3806, 51, 63, 218}

$$-\frac{3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\middle|\,-7-}{2d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(2/3),x]
```

```
[Out] (-3*a*Cos[c + d*x]*Csc[c + d*x]^(1/3))/(2*d*Sqrt[a + a*Csc[c + d*x]]) - (3^
(3/4)*Sqrt[2 + Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 +
 Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3)
)^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc
[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(2*d*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 +
 Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a + a*Csc[c + d
*x]])
```

Rule 51

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
((a + b*x)^(m + 1)*(c + d*x)^(n + 1))/((b*c - a*d)*(m + 1)), x] - Dist[(d*(
m + n + 2))/((b*c - a*d)*(m + 1)), Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x
] /; FreeQ[{a, b, c, d, n}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&  !(LtQ
[n, -1] && (EqQ[a, 0] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && I
ntLinearQ[a, b, c, d, m, n, x]
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx = \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{x^{5/3}\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)\sqrt[3]{\csc(c+dx)}}{2d\sqrt{a+a\csc(c+dx)}} + \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{x^{2/3}\sqrt{a-ax}}\,dx, x, \csc(c+dx\right.}{4d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)\sqrt[3]{\csc(c+dx)}}{2d\sqrt{a+a\csc(c+dx)}} + \frac{\left(3a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{\sqrt{a-ax^3}}\,dx, x, \sqrt[3]{\csc(c+d}\right.}{4d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)\sqrt[3]{\csc(c+dx)}}{2d\sqrt{a+a\csc(c+dx)}} - \frac{3^{3/4}\sqrt{2+\sqrt{3}}\,a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1}{}}}{2d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\,(a}$$

**Mathematica [C]** time = 0.46, size = 110, normalized size = 0.43

$$-\frac{\sqrt{a(\csc(c+dx)+1)}\left(\csc^{\frac{2}{3}}(c+dx)\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{3}{2};1-\csc(c+dx)\right)+3\right)\left(\cos\left(\frac{1}{2}(c+dx)\right)-\sin\left(\frac{1}{2}(c+dx)\right)\right)}{2d\csc^{\frac{2}{3}}(c+dx)\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(2/3),x]
```

```
[Out] -1/2*(Sqrt[a*(1 + Csc[c + d*x])]*(3 + Csc[c + d*x]^(2/3)*Hypergeometric2F1[
1/2, 2/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2]))/(d
*Csc[c + d*x]^(2/3)*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

**fricas [F]** time = 0.70, size = 0, normalized size = 0.00

$$\text{integral}\left(\frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{2}{3}}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x, algorithm="fricas")
```

```
[Out] integral(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(2/3), x)
```

**giac** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a \csc (dx + c) + a}}{\csc (dx + c)^{\frac{2}{3}}} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x, algorithm="giac")
[Out] integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(2/3), x)
```

**maple** [F] time = 2.98, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + a \csc (dx + c)}}{\csc (dx + c)^{\frac{2}{3}}} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x)
[Out] int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x)
```

**maxima** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a \csc (dx + c) + a}}{\csc (dx + c)^{\frac{2}{3}}} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x, algorithm="maxima")
[Out] integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(2/3), x)
```

**mupad** [F] time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{a + \frac{a}{\sin(c+d\,x)}}}{{\left(\frac{1}{\sin(c+d\,x)}\right)}^{2/3}} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(2/3),x)
[Out] int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(2/3), x)
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a (\csc (c + dx) + 1)}}{\csc^{\frac{2}{3}} (c + dx)} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))**(1/2)/csc(d*x+c)**(2/3),x)
[Out] Integral(sqrt(a*(csc(c + d*x) + 1))/csc(c + d*x)**(2/3), x)
```

## 3.24 $\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

**Optimal**. Leaf size=514

$$\frac{8\sqrt{2}\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4\sqrt{3}\right)}{7d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}\quad 12$$

```
[Out] -6/7*a*cos(d*x+c)*csc(d*x+c)^(5/3)/d/(a+a*csc(d*x+c))^(1/2)+24/7*a*cot(d*x+
c)/d/(1-csc(d*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)+8/7*3^(3/4)*a^2*co
t(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-csc
(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3)+csc(d*x+
c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc
(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)-
12/7*3^(1/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*x+c)^(1
/3)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)-1/2*2
^(1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))
^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(
1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

**Rubi [A]** time = 0.30, antiderivative size = 514, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.240$, Rules used = {3806, 50, 63, 303, 218, 1877}

$$\frac{8\sqrt{2}\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4\sqrt{3}\right)}{7d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}\quad 12$$

Antiderivative was successfully verified.

```
[In] Int[Csc[c + d*x]^(5/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (24*a*Cot[c + d*x])/(7*d*(1 + Sqrt[3] - Csc[c + d*x]^(1/3))*Sqrt[a + a*Csc[
c + d*x]]) - (6*a*Cos[c + d*x]*Csc[c + d*x]^(5/3))/(7*d*Sqrt[a + a*Csc[c +
d*x]]) - (12*3^(1/4)*Sqrt[2 - Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(
1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc
[c + d*x]^(1/3))^2]*EllipticE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1
+ Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(7*d*Sqrt[(1 - Csc[c + d
*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[
a + a*Csc[c + d*x]]) + (8*Sqrt[2]*3^(3/4)*a^2*Cot[c + d*x]*(1 - Csc[c + d*x
]^(1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] -
Csc[c + d*x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/
(1 + Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(7*d*Sqrt[(1 - Csc[c
+ d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sq
rt[a + a*Csc[c + d*x]])
```

Rule 50

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
((a + b*x)^(m + 1)*(c + d*x)^n)/(b*(m + n + 1)), x] + Dist[(n*(b*c - a*d))/
(b*(m + n + 1)), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 303

```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Dist[(Sqrt[2]*s)/(Sqrt[2 + Sqrt[3]]*r), Int[1/Sq
rt[a + b*x^3], x], x] + Dist[1/r, Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^
3], x], x]] /; FreeQ[{a, b}, x] && PosQ[a]
```

Rule 1877

```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[((1 - Sqrt[3])*d)/c]], s = Denom[Simplify[((1 - Sqrt[3])*d)/c
]]}, Simp[(2*d*s^3*Sqrt[a + b*x^3])/(a*r^2*((1 + Sqrt[3])*s + r*x)), x] - S
imp[(3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sq
rt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(r^2*Sqrt[a + b*x^3]*Sqrt[(s*(s + r*x))/
((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{x^{2/3}}{\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{6a\cos(c+dx)\csc^{\frac{5}{3}}(c+dx)}{7d\sqrt{a+a\csc(c+dx)}} + \frac{\left(4a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{\sqrt[3]{x}\sqrt{a-ax}}\,dx,\right.}{7d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(}$$

$$= -\frac{6a\cos(c+dx)\csc^{\frac{5}{3}}(c+dx)}{7d\sqrt{a+a\csc(c+dx)}} + \frac{\left(12a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{x}{\sqrt{a-ax^3}}\,dx, x\right.}{7d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc}}$$

$$= -\frac{6a\cos(c+dx)\csc^{\frac{5}{3}}(c+dx)}{7d\sqrt{a+a\csc(c+dx)}} - \frac{\left(12a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1-\sqrt{3}-x}{\sqrt{a-ax^3}}\,dx, x\right.}{7d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc}}$$

$$= \frac{24a\cot(c+dx)}{7d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}} - \frac{6a\cos(c+dx)\csc^{\frac{5}{3}}(c}{7d\sqrt{a+a\csc(c+}}$$

**Mathematica [C]** time = 1.18, size = 120, normalized size = 0.23

$$-\frac{2\sqrt{a(\csc(c+dx)+1)}\left(3(\csc(c+dx)+4)-8\sqrt[3]{\csc(c+dx)}\,{}_2F_1\left(\frac{1}{2},\frac{4}{3};\frac{3}{2};1-\csc(c+dx)\right)\right)\left(\cos\left(\frac{1}{2}(c+dx)\right)-\sin\right.}{7d\sqrt[3]{\csc(c+dx)}\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^(5/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (-2*Sqrt[a*(1 + Csc[c + d*x])]*(3*(4 + Csc[c + d*x]) - 8*Csc[c + d*x]^(1/3)
*Hypergeometric2F1[1/2, 4/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Si
n[(c + d*x)/2]))/(7*d*Csc[c + d*x]^(1/3)*(Cos[(c + d*x)/2] + Sin[(c + d*x)/
2]))
```

**fricas [F]** time = 0.89, size = 0, normalized size = 0.00

$$\text{integral}\left(\sqrt{a\csc(dx+c)+a}\,\csc(dx+c)^{\frac{5}{3}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

```
[Out] integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(5/3), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a\csc(dx+c)+a}\,\csc(dx+c)^{\frac{5}{3}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(5/3), x)
```

**maple [F]** time = 2.41, size = 0, normalized size = 0.00

$$\int \left(\csc^{\frac{5}{3}}(dx+c)\right)\sqrt{a+a\csc(dx+c)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x)
```

```
[Out] int(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc\left(dx + c\right) + a} \csc\left(dx + c\right)^{\frac{5}{3}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(5/3), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.00

$$\int \sqrt{a+\frac{a}{\sin\left(c+d\,x\right)}}\,{\left(\frac{1}{\sin\left(c+d\,x\right)}\right)}^{5/3} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(5/3),x)
```

```
[Out] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(5/3), x)
```

**sympy [F(-1)]** time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**(5/3)*(a+a*csc(d*x+c))**(1/2),x)
```

```
[Out] Timed out
```

## 3.25 $\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

**Optimal**. Leaf size=470

$$\frac{2\sqrt{2}\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4\sqrt{3}\right)\ 3\sqrt[4]{}}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

```
[Out] 6*a*cot(d*x+c)/d/(1-csc(d*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)+2*3^(3
/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/
2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3
)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c)
)/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))
^2)^(1/2)-3*3^(1/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*
x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2
)-1/2*2^(1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3
^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(
1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

**Rubi [A]** time = 0.25, antiderivative size = 470, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {3806, 63, 303, 218, 1877}

$$\frac{2\sqrt{2}\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4\sqrt{3}\right)\ 3\sqrt[4]{}}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[c + d*x]^(2/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (6*a*Cot[c + d*x])/(d*(1 + Sqrt[3] - Csc[c + d*x]^(1/3))*Sqrt[a + a*Csc[c +
 d*x]]) - (3*3^(1/4)*Sqrt[2 - Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(
1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc
[c + d*x]^(1/3))^2]*EllipticE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1
+ Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(d*Sqrt[(1 - Csc[c + d*x
]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a
+ a*Csc[c + d*x]]) + (2*Sqrt[2]*3^(3/4)*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^
(1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Cs
c[c + d*x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1
 + Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(d*Sqrt[(1 - Csc[c + d*
x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a
 + a*Csc[c + d*x]])
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
```

```
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 303

```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Dist[(Sqrt[2]*s)/(Sqrt[2 + Sqrt[3]]*r), Int[1/Sq
rt[a + b*x^3], x], x] + Dist[1/r, Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^
3], x], x]] /; FreeQ[{a, b}, x] && PosQ[a]
```

Rule 1877

```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[((1 - Sqrt[3])*d)/c]], s = Denom[Simplify[((1 - Sqrt[3])*d)/c
]]}, Simp[(2*d*s^3*Sqrt[a + b*x^3])/(a*r^2*((1 + Sqrt[3])*s + r*x)), x] - S
imp[(3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sq
rt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(r^2*Sqrt[a + b*x^3]*Sqrt[(s*(s + r*x))/
((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{\sqrt[3]{x}\,\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}}$$

$$= \frac{\left(3a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{x}{\sqrt{a-ax^3}}\,dx, x, \sqrt[3]{\csc(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{\left(3a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1-\sqrt{3}-x}{\sqrt{a-ax^3}}\,dx, x, \sqrt[3]{\csc(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\,\sqrt{a+a\csc(c+dx)}} - \frac{\left(3\sqrt{2}\left(2\right.\right.}{}$$

$$= \frac{6a\cot(c+dx)}{d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}} - \frac{3\sqrt[4]{3}\sqrt{2-\sqrt{3}}\,a^2\,\text{c}}{}$$

**Mathematica [C]** time = 0.97, size = 109, normalized size = 0.23

$$\frac{2\sqrt{a(\csc(c+dx)+1)}\left(2\sqrt[3]{\csc(c+dx)}\,{}_2F_1\left(\frac{1}{2},\frac{4}{3};\frac{3}{2};1-\csc(c+dx)\right)-3\right)\left(\cos\left(\frac{1}{2}(c+dx)\right)-\sin\left(\frac{1}{2}(c+dx)\right)\right)}{d\sqrt[3]{\csc(c+dx)}\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^(2/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (2*Sqrt[a*(1 + Csc[c + d*x])]*(-3 + 2*Csc[c + d*x]^(1/3)*Hypergeometric2F1[
1/2, 4/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2]))/(d
*Csc[c + d*x]^(1/3)*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

**fricas [F]** time = 1.45, size = 0, normalized size = 0.00

$$\text{integral}\left(\sqrt{a \csc\left(dx + c\right) + a} \csc\left(dx + c\right)^{\frac{2}{3}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

```
[Out] integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(2/3), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc\left(dx + c\right) + a} \csc\left(dx + c\right)^{\frac{2}{3}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(2/3), x)
```

**maple [F]** time = 2.18, size = 0, normalized size = 0.00

$$\int \left(\csc^{\frac{2}{3}}\left(dx + c\right)\right) \sqrt{a + a \csc\left(dx + c\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x)
```

```
[Out] int(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc\left(dx + c\right) + a} \csc\left(dx + c\right)^{\frac{2}{3}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(2/3), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.00

$$\int \sqrt{a+\frac{a}{\sin\left(c+d\,x\right)}}\,{\left(\frac{1}{\sin\left(c+d\,x\right)}\right)}^{2/3} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(2/3),x)
```

```
[Out] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(2/3), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \left(\csc{\left(c + dx \right)} + 1\right)} \csc^{\frac{2}{3}}{\left(c + dx \right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**(2/3)*(a+a*csc(d*x+c))**(1/2),x)
```

```
[Out] Integral(sqrt(a*(csc(c + d*x) + 1))*csc(c + d*x)**(2/3), x)
```

## 3.26 $\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx$

**Optimal**. Leaf size=508

$$-\frac{\sqrt{2}\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\middle|-7-4\sqrt{3}\right)}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}+$$

```
[Out] -3*a*cos(d*x+c)*csc(d*x+c)^(2/3)/d/(a+a*csc(d*x+c))^(1/2)-3*a*cot(d*x+c)/d/
(1-csc(d*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)-3^(3/4)*a^2*cot(d*x+c)*
(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c)^(
1/3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))
/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^
(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)+3/2*3^(1/
4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*x+c)^(1/3)-3^(1/2
))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)-1/2*2^(1/2))*((
1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/
d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+
c)^(1/3)+3^(1/2))^2)^(1/2)
```

**Rubi [A]** time = 0.27, antiderivative size = 508, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.240$, Rules used = {3806, 51, 63, 303, 218, 1877}

$$-\frac{\sqrt{2}\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\middle|-7-4\sqrt{3}\right)}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}+$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(1/3),x]
```

```
[Out] (-3*a*Cot[c + d*x])/(d*(1 + Sqrt[3] - Csc[c + d*x]^(1/3))*Sqrt[a + a*Csc[c
+ d*x]]) - (3*a*Cos[c + d*x]*Csc[c + d*x]^(2/3))/(d*Sqrt[a + a*Csc[c + d*x]
]) + (3*3^(1/4)*Sqrt[2 - Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3))
*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c +
d*x]^(1/3))^2]*EllipticE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqr
t[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(2*d*Sqrt[(1 - Csc[c + d*x]^(
1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a + a
*Csc[c + d*x]]) - (Sqrt[2]*3^(3/4)*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3)
)*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c +
 d*x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sq
rt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(d*Sqrt[(1 - Csc[c + d*x]^(1
/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a + a*
Csc[c + d*x]])
```

Rule 51

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
((a + b*x)^(m + 1)*(c + d*x)^(n + 1))/((b*c - a*d)*(m + 1)), x] - Dist[(d*(
m + n + 2))/((b*c - a*d)*(m + 1)), Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x
] /; FreeQ[{a, b, c, d, n}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&  !(LtQ
[n, -1] && (EqQ[a, 0] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && I
ntLinearQ[a, b, c, d, m, n, x]
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 303

```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Dist[(Sqrt[2]*s)/(Sqrt[2 + Sqrt[3]]*r), Int[1/Sq
rt[a + b*x^3], x], x] + Dist[1/r, Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^
3], x], x]] /; FreeQ[{a, b}, x] && PosQ[a]
```

Rule 1877

```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[((1 - Sqrt[3])*d)/c]], s = Denom[Simplify[((1 - Sqrt[3])*d)/c
]]}, Simp[(2*d*s^3*Sqrt[a + b*x^3])/(a*r^2*((1 + Sqrt[3])*s + r*x)), x] - S
imp[(3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sq
rt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(r^2*Sqrt[a + b*x^3]*Sqrt[(s*(s + r*x))/
((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx = \frac{\left(a^2\cot(c+dx)\right)\operatorname{Subst}\left(\int \frac{1}{x^{4/3}\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{d\sqrt{a+a\csc(c+dx)}} - \frac{\left(a^2\cot(c+dx)\right)\operatorname{Subst}\left(\int \frac{1}{\sqrt[3]{x}\sqrt{a-ax}}\,dx, x, \csc(c+\right.}{2d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{d\sqrt{a+a\csc(c+dx)}} - \frac{\left(3a^2\cot(c+dx)\right)\operatorname{Subst}\left(\int \frac{x}{\sqrt{a-ax^3}}\,dx, x, \sqrt[3]{\csc(c+}\right.}{2d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{d\sqrt{a+a\csc(c+dx)}} + \frac{\left(3a^2\cot(c+dx)\right)\operatorname{Subst}\left(\int \frac{1-\sqrt{3}-x}{\sqrt{a-ax^3}}\,dx, x, \sqrt[3]{\csc(c-}\right.}{2d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cot(c+dx)}{d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}} - \frac{3a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{d\sqrt{a+a\csc(c+dx)}} +$$

**Mathematica [C]** time = 0.76, size = 46, normalized size = 0.09

$$-\frac{2a\cot(c+dx)\,{}_2F_1\left(\frac{1}{2},\frac{4}{3};\frac{3}{2};1-\csc(c+dx)\right)}{d\sqrt{a(\csc(c+dx)+1)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(1/3),x]
```

```
[Out] (-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 4/3, 3/2, 1 - Csc[c + d*x]])/(d*S
qrt[a*(1 + Csc[c + d*x])])
```

**fricas [F]** time = 1.65, size = 0, normalized size = 0.00

$$\operatorname{integral}\left(\frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{1}{3}}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x, algorithm="fricas")
```

```
[Out] integral(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(1/3), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{1}{3}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x, algorithm="giac")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(1/3), x)
```

**maple [F]** time = 2.68, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a+a\csc(dx+c)}}{\csc(dx+c)^{\frac{1}{3}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x)
```

```
[Out] int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a \csc\left(dx + c\right) + a}}{\csc\left(dx + c\right)^{\frac{1}{3}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(1/3), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{a+\frac{a}{\sin(c+d\,x)}}}{{\left(\frac{1}{\sin(c+d\,x)}\right)}^{1/3}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(1/3),x)
```

```
[Out] int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(1/3), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a\left(\csc\left(c + dx\right) + 1\right)}}{\sqrt[3]{\csc\left(c + dx\right)}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))**(1/2)/csc(d*x+c)**(1/3),x)
```

```
[Out] Integral(sqrt(a*(csc(c + d*x) + 1))/csc(c + d*x)**(1/3), x)
```

## 3.27 $\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx$

**Optimal**. Leaf size=552

$$-\frac{5\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4\sqrt{3}\right)\ 1}{4\sqrt{2}\,d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}+\cdots$$

```
[Out] -3/4*a*cos(d*x+c)/d/csc(d*x+c)^(1/3)/(a+a*csc(d*x+c))^(1/2)-15/8*a*cos(d*x+
c)*csc(d*x+c)^(2/3)/d/(a+a*csc(d*x+c))^(1/2)-15/8*a*cot(d*x+c)/d/(1-csc(d*x
+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)-5/8*3^(3/4)*a^2*cot(d*x+c)*(1-csc
(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3
^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-cs
c(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/
((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)+15/16*3^(1/4)*a
^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*x+c)^(1/3)-3^(1/2))/(
1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)-1/2*2^(1/2))*((1+cs
c(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a
-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(
1/3)+3^(1/2))^2)^(1/2)
```

**Rubi [A]** time = 0.31, antiderivative size = 552, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.240, Rules used = {3806, 51, 63, 303, 218, 1877}

$$-\frac{5\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,F\left(\sin^{-1}\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)|-7-4\sqrt{3}\right)\ 1}{4\sqrt{2}\,d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}+\cdots$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(4/3),x]
```

```
[Out] (-15*a*Cot[c + d*x])/(8*d*(1 + Sqrt[3] - Csc[c + d*x]^(1/3))*Sqrt[a + a*Csc
[c + d*x]]) - (3*a*Cos[c + d*x])/(4*d*Csc[c + d*x]^(1/3)*Sqrt[a + a*Csc[c +
 d*x]]) - (15*a*Cos[c + d*x]*Csc[c + d*x]^(2/3))/(8*d*Sqrt[a + a*Csc[c + d*
x]]) + (15*3^(1/4)*Sqrt[2 - Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/
3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c
 + d*x]^(1/3))^2]*EllipticE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 +
Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(16*d*Sqrt[(1 - Csc[c + d*
x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a
 + a*Csc[c + d*x]]) - (5*3^(3/4)*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3))*
Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d
*x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt
[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(4*Sqrt[2]*d*Sqrt[(1 - Csc[c +
 d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqr
t[a + a*Csc[c + d*x]])
```

Rule 51

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
((a + b*x)^(m + 1)*(c + d*x)^(n + 1))/((b*c - a*d)*(m + 1)), x] - Dist[(d*(
m + n + 2))/((b*c - a*d)*(m + 1)), Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x
] /; FreeQ[{a, b, c, d, n}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&  !(LtQ
```

```
[n, -1] && (EqQ[a, 0] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && I
ntLinearQ[a, b, c, d, m, n, x]
```

Rule 63

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - (a*d)/b +
(d*x^p)/b)^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 218

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[(2*Sqrt[2 + Sqrt[3]]*(s + r*x)*Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(3^(1/4)*r*Sqrt[a + b*x^3
]*Sqrt[(s*(s + r*x))/((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

Rule 303

```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Dist[(Sqrt[2]*s)/(Sqrt[2 + Sqrt[3]]*r), Int[1/Sq
rt[a + b*x^3], x], x] + Dist[1/r, Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^
3], x], x]] /; FreeQ[{a, b}, x] && PosQ[a]
```

Rule 1877

```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[((1 - Sqrt[3])*d)/c]], s = Denom[Simplify[((1 - Sqrt[3])*d)/c
]]}, Simp[(2*d*s^3*Sqrt[a + b*x^3])/(a*r^2*((1 + Sqrt[3])*s + r*x)), x] - S
imp[(3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sq
rt[3])*s + r*x)], -7 - 4*Sqrt[3]])/(r^2*Sqrt[a + b*x^3]*Sqrt[(s*(s + r*x))/
((1 + Sqrt[3])*s + r*x)^2]), x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx = \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{x^{7/3}\sqrt{a-ax}}\,dx, x, \csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}$$

$$= -\frac{3a\cos(c+dx)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}} + \frac{\left(5a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{1}{x^{4/3}\sqrt{a-ax}}\,dx, x\right.}{8d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c}$$

$$= -\frac{3a\cos(c+dx)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}} - \frac{15a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{8d\sqrt{a+a\csc(c+dx)}} - \frac{\left(5a^2\cot(c\right.}{16a}$$

$$= -\frac{3a\cos(c+dx)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}} - \frac{15a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{8d\sqrt{a+a\csc(c+dx)}} - \frac{\left(15a^2\cot\right.}{16}$$

$$= -\frac{3a\cos(c+dx)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}} - \frac{15a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{8d\sqrt{a+a\csc(c+dx)}} + \frac{\left(15a^2\cot\right.}{16}$$

$$= -\frac{15a\cot(c+dx)}{8d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}} - \frac{3a\cos(c+dx)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c}$$

**Mathematica [C]** time = 1.22, size = 72, normalized size = 0.13

$$-\frac{a\cos(c+dx)\left(5\csc^{\frac{4}{3}}(c+dx)\,{}_2F_1\left(\frac{1}{2},\frac{4}{3};\frac{3}{2};1-\csc(c+dx)\right)+3\right)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a(\csc(c+dx)+1)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(4/3),x]
```

```
[Out] -1/4*(a*Cos[c + d*x]*(3 + 5*Csc[c + d*x]^(4/3)*Hypergeometric2F1[1/2, 4/3,
3/2, 1 - Csc[c + d*x]]))/(d*Csc[c + d*x]^(1/3)*Sqrt[a*(1 + Csc[c + d*x])])
```

**fricas [F]** time = 1.89, size = 0, normalized size = 0.00

$$\text{integral}\left(\frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{4}{3}}}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x, algorithm="fricas")
```

```
[Out] integral(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(4/3), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{4}{3}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x, algorithm="giac")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(4/3), x)
```

**maple [F]** time = 1.80, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + a\csc\left(dx + c\right)}}{\csc\left(dx + c\right)^{\frac{4}{3}}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x)
```

```
[Out] int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a\csc\left(dx + c\right) + a}}{\csc\left(dx + c\right)^{\frac{4}{3}}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(4/3), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{a + \frac{a}{\sin(c+d\,x)}}}{\left(\frac{1}{\sin(c+d\,x)}\right)^{4/3}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(4/3),x)
```

```
[Out] int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(4/3), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a\left(\csc\left(c + dx\right) + 1\right)}}{\csc^{\frac{4}{3}}\left(c + dx\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))**(1/2)/csc(d*x+c)**(4/3),x)
```

```
[Out] Integral(sqrt(a*(csc(c + d*x) + 1))/csc(c + d*x)**(4/3), x)
```

## 3.28 $\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

**Optimal**. Leaf size=48

$$-\frac{2a\cot(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};1-\csc(c+dx)\right)}{d\sqrt{a\csc(c+dx)+a}}$$

```
[Out] -2*a*cot(d*x+c)*hypergeom([1/2, 1-n],[3/2],1-csc(d*x+c))/d/(a+a*csc(d*x+c))
^(1/2)
```

**Rubi** [A] time = 0.06, antiderivative size = 48, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 23, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.087, Rules used = {3806, 65}

$$-\frac{2a\cot(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};1-\csc(c+dx)\right)}{d\sqrt{a\csc(c+dx)+a}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[c + d*x]^n*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 1 - n, 3/2, 1 - Csc[c + d*x]])/(d
*Sqrt[a + a*Csc[c + d*x]])
```

Rule 65

```
Int[((b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((c + d*x
)^(n + 1)*Hypergeometric2F1[-m, n + 1, n + 2, 1 + (d*x)/c])/(d*(n + 1)*(-(d
/(b*c)))^m), x] /; FreeQ[{b, c, d, m, n}, x] &&  !IntegerQ[n] && (IntegerQ[
m] || GtQ[-(d/(b*c)), 0])
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\begin{aligned}\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx &= \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{x^{-1+n}}{\sqrt{a-ax}}\,dx,x,\csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}\\ &= -\frac{2a\cot(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};1-\csc(c+dx)\right)}{d\sqrt{a+a\csc(c+dx)}}\end{aligned}$$

**Mathematica** [A] time = 0.17, size = 48, normalized size = 1.00

$$-\frac{2a\cot(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};1-\csc(c+dx)\right)}{d\sqrt{a(\csc(c+dx)+1)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^n*Sqrt[a + a*Csc[c + d*x]],x]
```

```
[Out] (-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 1 - n, 3/2, 1 - Csc[c + d*x]])/(d
*Sqrt[a*(1 + Csc[c + d*x])])
```

**fricas [F]** time = 0.94, size = 0, normalized size = 0.00

$$\text{integral}\left(\sqrt{a \csc(dx+c) + a} \csc(dx+c)^{n}, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
[Out] integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc(dx+c) + a} \csc(dx+c)^{n} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

**maple [F]** time = 1.95, size = 0, normalized size = 0.00

$$\int (\csc^{n}(dx+c)) \sqrt{a + a \csc(dx+c)} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x)
[Out] int(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a \csc(dx+c) + a} \csc(dx+c)^{n} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
[Out] integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.02

$$\int \sqrt{a + \frac{a}{\sin(c + d\,x)}} \left(\frac{1}{\sin(c + d\,x)}\right)^{n} dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n,x)
[Out] int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n, x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{a (\csc(c + dx) + 1)} \csc^{n}(c + dx) \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**n*(a+a*csc(d*x+c))**(1/2),x)
[Out] Integral(sqrt(a*(csc(c + d*x) + 1))*csc(c + d*x)**n, x)
```

## 3.29 $\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx$

**Optimal**. Leaf size=69

$$-\frac{2a\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{n+1}(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\csc(c+dx)+1\right)}{d\sqrt{a-a\csc(c+dx)}}$$

```
[Out] -2*a*cos(d*x+c)*csc(d*x+c)^(1+n)*hypergeom([1/2, 1-n],[3/2],1+csc(d*x+c))/d
/((-csc(d*x+c))^n)/(a-a*csc(d*x+c))^(1/2)
```

**Rubi [A]** time = 0.07, antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.125, Rules used = {3806, 67, 65}

$$-\frac{2a\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{n+1}(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\csc(c+dx)+1\right)}{d\sqrt{a-a\csc(c+dx)}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[c + d*x]^n*Sqrt[a - a*Csc[c + d*x]],x]
```

```
[Out] (-2*a*Cos[c + d*x]*Csc[c + d*x]^(1 + n)*Hypergeometric2F1[1/2, 1 - n, 3/2,
1 + Csc[c + d*x]])/(d*(-Csc[c + d*x])^n*Sqrt[a - a*Csc[c + d*x]])
```

Rule 65

```
Int[((b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((c + d*x
)^(n + 1)*Hypergeometric2F1[-m, n + 1, n + 2, 1 + (d*x)/c])/(d*(n + 1)*(-(d
/(b*c)))^m), x] /; FreeQ[{b, c, d, m, n}, x] &&  !IntegerQ[n] && (IntegerQ[
m] || GtQ[-(d/(b*c)), 0])
```

Rule 67

```
Int[((b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Dist[((-((b*c)
/d))^IntPart[m]*(b*x)^FracPart[m])/(-((d*x)/c))^FracPart[m], Int[(-((d*x)/c
))^m*(c + d*x)^n, x], x] /; FreeQ[{b, c, d, m, n}, x] &&  !IntegerQ[m] &&
!IntegerQ[n] &&  !GtQ[c, 0] &&  !GtQ[-(d/(b*c)), 0]
```

Rule 3806

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*x]
]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

Rubi steps

$$\begin{aligned}
\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx &= \frac{\left(a^2\cot(c+dx)\right)\text{Subst}\left(\int \frac{x^{-1+n}}{\sqrt{a+ax}}\,dx,x,\csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}\\
&= -\frac{\left(a^2\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{1+n}(c+dx)\right)\text{Subst}\left(\int \frac{(-x)^{-1+n}}{\sqrt{a+ax}}\,dx\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a+a\csc(c+dx)}}\\
&= -\frac{2a\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{1+n}(c+dx)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};1+\text{c}\right.}{d\sqrt{a-a\csc(c+dx)}}
\end{aligned}$$

**Mathematica** [A] time = 0.99, size = 73, normalized size = 1.06

$$-\frac{2a\cos(c+dx)\csc^{2n+1}(c+dx)\left(-\csc^2(c+dx)\right)^{-n}{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\csc(c+dx)+1\right)}{d\sqrt{a-a\csc(c+dx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^n*Sqrt[a - a*Csc[c + d*x]],x]
```

```
[Out] (-2*a*Cos[c + d*x]*Csc[c + d*x]^(1 + 2*n)*Hypergeometric2F1[1/2, 1 - n, 3/2
, 1 + Csc[c + d*x]])/(d*(-Csc[c + d*x]^2)^n*Sqrt[a - a*Csc[c + d*x]])
```

**fricas** [F] time = 0.65, size = 0, normalized size = 0.00

$$\text{integral}\left(\sqrt{-a\csc(dx+c)+a}\csc(dx+c)^n, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

```
[Out] integral(sqrt(-a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

**giac** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int\sqrt{-a\csc(dx+c)+a}\csc(dx+c)^n\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

```
[Out] integrate(sqrt(-a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

**maple** [F] time = 2.45, size = 0, normalized size = 0.00

$$\int(\csc^n(dx+c))\sqrt{a-a\csc(dx+c)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x)
```

```
[Out] int(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x)
```

**maxima** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int\sqrt{-a\csc(dx+c)+a}\csc(dx+c)^n\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(-a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

**mupad** [F] time = 0.00, size = -1, normalized size = -0.01

$$\int\sqrt{a-\frac{a}{\sin(c+d\,x)}}\left(\frac{1}{\sin(c+d\,x)}\right)^n\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a - a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n,x)
```

```
[Out] int((a - a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n, x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{-a\left(\csc\left(c+dx\right)-1\right)}\,\csc^{n}\left(c+dx\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**n*(a-a*csc(d*x+c))**(1/2),x)
[Out] Integral(sqrt(-a*(csc(c + d*x) - 1))*csc(c + d*x)**n, x)
```

# 3.30 $\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx$

**Optimal**. Leaf size=156

$$-\frac{2^{m+\frac{1}{2}}\left(m^2+m+1\right)\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx))\right)}{f(m+1)(m+2)}+$$

```
[Out] cot(f*x+e)*(a+a*csc(f*x+e))^m/f/(m^2+3*m+2)-cot(f*x+e)*(a+a*csc(f*x+e))^(1+
m)/a/f/(2+m)-2^(1/2+m)*(m^2+m+1)*cot(f*x+e)*(1+csc(f*x+e))^(-1/2-m)*(a+a*cs
c(f*x+e))^m*hypergeom([1/2, 1/2-m],[3/2],1/2-1/2*csc(f*x+e))/f/(m^2+3*m+2)
```

**Rubi [A]** time = 0.19, antiderivative size = 156, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 21, $\frac{\text{number of rules}}{\text{integrand size}} = 0.238$, Rules used = {3800, 4001, 3828, 3827, 69}

$$-\frac{2^{m+\frac{1}{2}}\left(m^2+m+1\right)\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx))\right)}{f(m+1)(m+2)}+$$

Antiderivative was successfully verified.

```
[In] Int[Csc[e + f*x]^3*(a + a*Csc[e + f*x])^m,x]
```

```
[Out] (Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(2 + 3*m + m^2)) - (Cot[e + f*x]*(
a + a*Csc[e + f*x])^(1 + m))/(a*f*(2 + m)) - (2^(1/2 + m)*(1 + m + m^2)*Cot
[e + f*x]*(1 + Csc[e + f*x])^(-1/2 - m)*(a + a*Csc[e + f*x])^m*Hypergeometr
ic2F1[1/2, 1/2 - m, 3/2, (1 - Csc[e + f*x])/2])/(f*(1 + m)*(2 + m))
```

Rule 69

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((
a + b*x)^(m + 1)*Hypergeometric2F1[-n, m + 1, m + 2, -((d*(a + b*x))/(b*c -
 a*d))])/(b*(m + 1)*(b/(b*c - a*d))^n), x] /; FreeQ[{a, b, c, d, m, n}, x]
&& NeQ[b*c - a*d, 0] && !IntegerQ[m] && !IntegerQ[n] && GtQ[b/(b*c - a*d)
, 0] && (RationalQ[m] || !(RationalQ[n] && GtQ[-(d/(b*c - a*d)), 0]))
```

Rule 3800

```
Int[csc[(e_.) + (f_.)*(x_)]^3*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> -Simp[(Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1))/(b*f*(m + 2)
), x] + Dist[1/(b*(m + 2)), Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*(b*(m +
 1) - a*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f, m}, x] && EqQ[a^2 - b^2
, 0] && !LtQ[m, -2^(-1)]
```

Rule 3827

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*
x]]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[((d*x)^(n - 1)*(a + b*x)^(m - 1/2)
)/Sqrt[a - b*x], x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n}, x
] && EqQ[a^2 - b^2, 0] && !IntegerQ[m] && GtQ[a, 0]
```

Rule 3828

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^IntPart[m]*(a + b*Csc[e + f*x])^FracPart[m
])/(1 + (b*Csc[e + f*x])/a)^FracPart[m], Int[(1 + (b*Csc[e + f*x])/a)^m*(d*
Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^2
, 0] && !IntegerQ[m] && !GtQ[a, 0]
```

Rule 4001

```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(cs
c[(e_.) + (f_.)*(x_)]*(B_.) + (A_)), x_Symbol] :> -Simp[(B*Cot[e + f*x]*(a
+ b*Csc[e + f*x])^m)/(f*(m + 1)), x] + Dist[(a*B*m + A*b*(m + 1))/(b*(m + 1
)), Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] /; FreeQ[{a, b, A, B, e
, f, m}, x] && NeQ[A*b - a*B, 0] && EqQ[a^2 - b^2, 0] && NeQ[a*B*m + A*b*(m
 + 1), 0] && !LtQ[m, -2^(-1)]
```

Rubi steps

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx = -\frac{\cot(e+fx)(a+a\csc(e+fx))^{1+m}}{af(2+m)} + \frac{\int \csc(e+fx)(a(1+m)-a\csc\cdots}{a(2\cdots}$$

$$= \frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f\left(2+3m+m^2\right)} - \frac{\cot(e+fx)(a+a\csc(e+fx))^{1+m}}{af(2+m)}\cdots$$

$$= \frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f\left(2+3m+m^2\right)} - \frac{\cot(e+fx)(a+a\csc(e+fx))^{1+m}}{af(2+m)}\cdots$$

$$= \frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f\left(2+3m+m^2\right)} - \frac{\cot(e+fx)(a+a\csc(e+fx))^{1+m}}{af(2+m)}\cdots$$

$$= \frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f\left(2+3m+m^2\right)} - \frac{\cot(e+fx)(a+a\csc(e+fx))^{1+m}}{af(2+m)}\cdots$$

**Mathematica [A]** time = 4.61, size = 178, normalized size = 1.14

$$-\frac{\tan^2\left(\frac{1}{2}(e+fx)\right)\left(\tan\left(\frac{1}{2}(e+fx)\right)+1\right)^{-2m}(a(\csc(e+fx)+1))^m\left((m-2)m\cot^4\left(\frac{1}{2}(e+fx)\right){}_2F_1\left(-m-2,-2\cdots\right.\right.}{\cdots}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[e + f*x]^3*(a + a*Csc[e + f*x])^m,x]
```

```
[Out] -1/4*((a*(1 + Csc[e + f*x]))^m*((-2 + m)*m*Cot[(e + f*x)/2]^4*Hypergeometri
c2F1[-2 - m, -2*m, -1 - m, -Tan[(e + f*x)/2]] + (2 + m)*(m*Hypergeometric2F
1[2 - m, -2*m, 3 - m, -Tan[(e + f*x)/2]] + 2*(-2 + m)*Cot[(e + f*x)/2]^2*Hy
pergeometric2F1[-2*m, -m, 1 - m, -Tan[(e + f*x)/2]]))*Tan[(e + f*x)/2]^2)/(
f*(-2 + m)*m*(2 + m)*(1 + Tan[(e + f*x)/2])^(2*m))
```

**fricas [F]** time = 0.53, size = 0, normalized size = 0.00

$$\mathrm{integral}\left(\left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^3, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x, algorithm="fricas")
```

```
[Out] integral((a*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^3\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

**maple [F]** time = 3.27, size = 0, normalized size = 0.00

$$\int \left(\csc^3\left(fx+e\right)\right)\left(a+a\csc\left(fx+e\right)\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x)
[Out] int(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m \csc\left(fx+e\right)^3\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x, algorithm="maxima")
[Out] integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\left(a+\frac{a}{\sin\left(e+f\,x\right)}\right)^m}{\sin\left(e+f\,x\right)^3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(e + f*x))^m/sin(e + f*x)^3,x)
[Out] int((a + a/sin(e + f*x))^m/sin(e + f*x)^3, x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\left(\csc\left(e+fx\right)+1\right)\right)^m \csc^3\left(e+fx\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)**3*(a+a*csc(f*x+e))**m,x)
[Out] Integral((a*(csc(e + f*x) + 1))**m*csc(e + f*x)**3, x)
```

## 3.31 $\int \csc^2(e+fx)(a+a\csc(e+fx))^m \, dx$

**Optimal**. Leaf size=109

$$-\frac{2^{m+\frac{1}{2}} m \cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m {}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx))\right)}{f(m+1)} - \cot(e+f$$

```
[Out] -cot(f*x+e)*(a+a*csc(f*x+e))^m/f/(1+m)-2^(1/2+m)*m*cot(f*x+e)*(1+csc(f*x+e)
)^(-1/2-m)*(a+a*csc(f*x+e))^m*hypergeom([1/2, 1/2-m],[3/2],1/2-1/2*csc(f*x+
e))/f/(1+m)
```

**Rubi [A]** time = 0.10, antiderivative size = 109, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 21, $\frac{\text{number of rules}}{\text{integrand size}} = 0.190$, Rules used = {3798, 3828, 3827, 69}

$$-\frac{2^{m+\frac{1}{2}} m \cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m {}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx))\right)}{f(m+1)} - \cot(e+f$$

Antiderivative was successfully verified.

```
[In] Int[Csc[e + f*x]^2*(a + a*Csc[e + f*x])^m,x]
```

```
[Out] -((Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + m))) - (2^(1/2 + m)*m*Cot[e
 + f*x]*(1 + Csc[e + f*x])^(-1/2 - m)*(a + a*Csc[e + f*x])^m*Hypergeometric
2F1[1/2, 1/2 - m, 3/2, (1 - Csc[e + f*x])/2])/(f*(1 + m))
```

Rule 69

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((
a + b*x)^(m + 1)*Hypergeometric2F1[-n, m + 1, m + 2, -((d*(a + b*x))/(b*c -
 a*d))])/(b*(m + 1)*(b/(b*c - a*d))^n), x] /; FreeQ[{a, b, c, d, m, n}, x]
&& NeQ[b*c - a*d, 0] && !IntegerQ[m] && !IntegerQ[n] && GtQ[b/(b*c - a*d)
, 0] && (RationalQ[m] || !(RationalQ[n] && GtQ[-(d/(b*c - a*d)), 0]))
```

Rule 3798

```
Int[csc[(e_.) + (f_.)*(x_)]^2*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> -Simp[(Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(f*(m + 1)), x] +
Dist[(a*m)/(b*(m + 1)), Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] /;
FreeQ[{a, b, e, f, m}, x] && EqQ[a^2 - b^2, 0] && !LtQ[m, -2^(-1)]
```

Rule 3827

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*
x]]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[((d*x)^(n - 1)*(a + b*x)^(m - 1/2)
)/Sqrt[a - b*x], x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n}, x
] && EqQ[a^2 - b^2, 0] && !IntegerQ[m] && GtQ[a, 0]
```

Rule 3828

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^IntPart[m]*(a + b*Csc[e + f*x])^FracPart[m
])/(1 + (b*Csc[e + f*x])/a)^FracPart[m], Int[(1 + (b*Csc[e + f*x])/a)^m*(d*
Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^2
, 0] && !IntegerQ[m] && !GtQ[a, 0]
```

Rubi steps

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = -\frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+m)} + \frac{m\int \csc(e+fx)(a+a\csc(e+fx))^{\cdots}}{1+m}$$

$$= -\frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+m)} + \frac{\left(m(1+\csc(e+fx))^{-m}(a+a\csc(e\cdots\right.}{\cdots}$$

$$= -\frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+m)} + \frac{\left(m\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-\cdots}\right.}{\cdots}$$

$$= -\frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+m)} - \frac{2^{\frac{1}{2}+m}m\cot(e+fx)(1+\csc(e+fx)\cdots}{\cdots}$$

**Mathematica** [A] time = 1.35, size = 126, normalized size = 1.16

$$-\frac{\tan\left(\frac{1}{2}(e+fx)\right)\left(\tan\left(\frac{1}{2}(e+fx)\right)+1\right)^{-2m}(a(\csc(e+fx)+1))^m\left((m+1)\,{}_2F_1\left(1-m,-2m;2-m;-\tan\left(\frac{1}{2}(e+f\cdots\right.\right.\right.}{2f(m-1)(m+1)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[e + f*x]^2*(a + a*Csc[e + f*x])^m,x]
[Out] -1/2*((a*(1 + Csc[e + f*x]))^m*((-1 + m)*Cot[(e + f*x)/2]^2*Hypergeometric2
F1[-1 - m, -2*m, -m, -Tan[(e + f*x)/2]] + (1 + m)*Hypergeometric2F1[1 - m,
-2*m, 2 - m, -Tan[(e + f*x)/2]])*Tan[(e + f*x)/2])/(f*(-1 + m)*(1 + m)*(1 +
 Tan[(e + f*x)/2])^(2*m))
```

**fricas** [F] time = 0.58, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^2,x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x, algorithm="fricas")
[Out] integral((a*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

**giac** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^2\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

**maple** [F] time = 2.00, size = 0, normalized size = 0.00

$$\int \left(\csc^2\left(fx+e\right)\right)\left(a+a\csc\left(fx+e\right)\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x)
[Out] int(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^{m}\csc\left(fx+e\right)^{2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

```
[Out] integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\left(a+\frac{a}{\sin\left(e+f\,x\right)}\right)^{m}}{{\sin\left(e+f\,x\right)}^{2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(e + f*x))^m/sin(e + f*x)^2,x)
```

```
[Out] int((a + a/sin(e + f*x))^m/sin(e + f*x)^2, x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\left(\csc\left(e+fx\right)+1\right)\right)^{m}\csc^{2}\left(e+fx\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)**2*(a+a*csc(f*x+e))**m,x)
```

```
[Out] Integral((a*(csc(e + f*x) + 1))**m*csc(e + f*x)**2, x)
```

# 3.32 $\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx$

**Optimal**. Leaf size=74

$$-\frac{2^{m+\frac{1}{2}}\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx))\right)}{f}$$

```
[Out] -2^(1/2+m)*cot(f*x+e)*(1+csc(f*x+e))^(-1/2-m)*(a+a*csc(f*x+e))^m*hypergeom(
[1/2, 1/2-m],[3/2],1/2-1/2*csc(f*x+e))/f
```

**Rubi [A]** time = 0.06, antiderivative size = 74, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 19, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.158, Rules used = {3828, 3827, 69}

$$-\frac{2^{m+\frac{1}{2}}\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx))\right)}{f}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[e + f*x]*(a + a*Csc[e + f*x])^m,x]
```

```
[Out] -((2^(1/2 + m)*Cot[e + f*x]*(1 + Csc[e + f*x])^(-1/2 - m)*(a + a*Csc[e + f*
x])^m*Hypergeometric2F1[1/2, 1/2 - m, 3/2, (1 - Csc[e + f*x])/2])/f)
```

Rule 69

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((
a + b*x)^(m + 1)*Hypergeometric2F1[-n, m + 1, m + 2, -((d*(a + b*x))/(b*c -
 a*d))])/(b*(m + 1)*(b/(b*c - a*d))^n), x] /; FreeQ[{a, b, c, d, m, n}, x]
&& NeQ[b*c - a*d, 0] &&  !IntegerQ[m] &&  !IntegerQ[n] && GtQ[b/(b*c - a*d)
, 0] && (RationalQ[m] ||  !(RationalQ[n] && GtQ[-(d/(b*c - a*d)), 0]))
```

Rule 3827

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*
x]]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[((d*x)^(n - 1)*(a + b*x)^(m - 1/2)
)/Sqrt[a - b*x], x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n}, x
] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

Rule 3828

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^IntPart[m]*(a + b*Csc[e + f*x])^FracPart[m
])/(1 + (b*Csc[e + f*x])/a)^FracPart[m], Int[(1 + (b*Csc[e + f*x])/a)^m*(d*
Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^2
, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

Rubi steps

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx = \left((1+\csc(e+fx))^{-m}(a+a\csc(e+fx))^m\right)\int \csc(e+fx)(1+\csc(e+$$

$$= \frac{\left(\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m\right)\text{Subst}\left(\int \frac{(1+x}{\sqrt{}}\right.}{f\sqrt{1-\csc(e+fx)}}$$

$$= -\frac{2^{\frac{1}{2}+m}\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}\right.}{f}$$

**Mathematica [A]** time = 0.20, size = 60, normalized size = 0.81

$$-\frac{\left(\tan\left(\frac{1}{2}(e+fx)\right)+1\right)^{-2m}(a(\csc(e+fx)+1))^m\,{}_2F_1\left(-2m,-m;1-m;-\tan\left(\frac{1}{2}(e+fx)\right)\right)}{fm}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[e + f*x]*(a + a*Csc[e + f*x])^m,x]
[Out] -(((a*(1 + Csc[e + f*x]))^m*Hypergeometric2F1[-2*m, -m, 1 - m, -Tan[(e + f*
x)/2]])/(f*m*(1 + Tan[(e + f*x)/2])^(2*m)))
```

**fricas [F]** time = 0.54, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right),x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+a*csc(f*x+e))^m,x, algorithm="fricas")
[Out] integral((a*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+a*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((a*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

**maple [F]** time = 1.54, size = 0, normalized size = 0.00

$$\int \csc\left(fx+e\right)\left(a+a\csc\left(fx+e\right)\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)*(a+a*csc(f*x+e))^m,x)
[Out] int(csc(f*x+e)*(a+a*csc(f*x+e))^m,x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

```
[Out] integrate((a*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\left(a+\frac{a}{\sin\left(e+f\,x\right)}\right)^m}{\sin\left(e+f\,x\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(e + f*x))^m/sin(e + f*x),x)
```

```
[Out] int((a + a/sin(e + f*x))^m/sin(e + f*x), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a \left(\csc\left(e+fx\right)+1\right)\right)^{m} \csc\left(e+fx\right)\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+a*csc(f*x+e))**m,x)
```

```
[Out] Integral((a*(csc(e + f*x) + 1))**m*csc(e + f*x), x)
```

## 3.33 $\int (a + a\csc(e + fx))^m\,dx$

**Optimal**. Leaf size=84

$$-\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m F_1\left(m+\frac{1}{2};\frac{1}{2},1;m+\frac{3}{2};\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

```
[Out] -AppellF1(1/2+m,1,1/2,3/2+m,1+csc(f*x+e),1/2+1/2*csc(f*x+e))*cot(f*x+e)*(a+
a*csc(f*x+e))^m*2^(1/2)/f/(1+2*m)/(1-csc(f*x+e))^(1/2)
```

**Rubi [A]** time = 0.06, antiderivative size = 84, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.250, Rules used = {3779, 3778, 136}

$$-\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m F_1\left(m+\frac{1}{2};\frac{1}{2},1;m+\frac{3}{2};\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[e + f*x])^m,x]
```

```
[Out] -((Sqrt[2]*AppellF1[1/2 + m, 1/2, 1, 3/2 + m, (1 + Csc[e + f*x])/2, 1 + Csc
[e + f*x]]*Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + 2*m)*Sqrt[1 - Csc[e
 + f*x]]))
```

Rule 136

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Simp[((b*e - a*f)^p*(a + b*x)^(m + 1)*AppellF1[m + 1, -
n, -p, m + 2, -((d*(a + b*x))/(b*c - a*d)), -((f*(a + b*x))/(b*e - a*f))])/
(b^(p + 1)*(m + 1)*(b/(b*c - a*d))^n), x] /; FreeQ[{a, b, c, d, e, f, m, n}
, x] && !IntegerQ[m] && !IntegerQ[n] && IntegerQ[p] && GtQ[b/(b*c - a*d),
 0] && !(GtQ[d/(d*a - c*b), 0] && SimplerQ[c + d*x, a + b*x])
```

Rule 3778

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Dist[(a^n*Cot
[c + d*x])/(d*Sqrt[1 + Csc[c + d*x]]*Sqrt[1 - Csc[c + d*x]]), Subst[Int[(1
+ (b*x)/a)^(n - 1/2)/(x*Sqrt[1 - (b*x)/a]), x], x, Csc[c + d*x]], x] /; Fre
eQ[{a, b, c, d, n}, x] && EqQ[a^2 - b^2, 0] && !IntegerQ[2*n] && GtQ[a, 0]
```

Rule 3779

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Dist[(a^IntPa
rt[n]*(a + b*Csc[c + d*x])^FracPart[n])/(1 + (b*Csc[c + d*x])/a)^FracPart[n
], Int[(1 + (b*Csc[c + d*x])/a)^n, x], x] /; FreeQ[{a, b, c, d, n}, x] && E
qQ[a^2 - b^2, 0] && !IntegerQ[2*n] && !GtQ[a, 0]
```

Rubi steps

$$\int (a+a\csc(e+fx))^m\,dx = \left((1+\csc(e+fx))^{-m}(a+a\csc(e+fx))^m\right)\int (1+\csc(e+fx))^m\,dx$$
$$= \frac{\left(\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m\right)\mathrm{Subst}\left(\int \frac{(1+x)^{-\frac{1}{2}+m}}{\sqrt{1-x}\,x}\,dx, x, \csc\right.}{f\sqrt{1-\csc(e+fx)}}$$
$$= -\frac{\sqrt{2}\,F_1\left(\frac{1}{2}+m;\frac{1}{2},1;\frac{3}{2}+m;\frac{1}{2}(1+\csc(e+fx)),1+\csc(e+fx)\right)\cot(e+fx)(a+a\,\mathrm{cs}}{f(1+2m)\sqrt{1-\csc(e+fx)}}$$

**Mathematica [F]** time = 0.61, size = 0, normalized size = 0.00

$$\int (a+a\csc(e+fx))^m\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[(a + a*Csc[e + f*x])^m,x]
[Out] Integrate[(a + a*Csc[e + f*x])^m, x]
```

**fricas [F]** time = 0.83, size = 0, normalized size = 0.00

$$\mathrm{integral}\left(\left(a\csc\left(fx+e\right)+a\right)^m, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m,x, algorithm="fricas")
[Out] integral((a*csc(f*x + e) + a)^m, x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((a*csc(f*x + e) + a)^m, x)
```

**maple [F]** time = 1.20, size = 0, normalized size = 0.00

$$\int \left(a+a\csc\left(fx+e\right)\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(f*x+e))^m,x)
[Out] int((a+a*csc(f*x+e))^m,x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

```
[Out] integrate((a*csc(f*x + e) + a)^m, x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \left(a+\frac{a}{\sin\left(e+f\,x\right)}\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + a/sin(e + f*x))^m,x)
[Out] int((a + a/sin(e + f*x))^m, x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a \csc \left(e+fx\right)+a\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))**m,x)
[Out] Integral((a*csc(e + f*x) + a)**m, x)
```

## 3.34 $\int (a + a\csc(e + fx))^m \sin(e + fx)\, dx$

**Optimal**. Leaf size=83

$$\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m F_1\left(m+\frac{1}{2};\frac{1}{2},2;m+\frac{3}{2};\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

```
[Out] AppellF1(1/2+m,2,1/2,3/2+m,1+csc(f*x+e),1/2+1/2*csc(f*x+e))*cot(f*x+e)*(a+a
*csc(f*x+e))^m*2^(1/2)/f/(1+2*m)/(1-csc(f*x+e))^(1/2)
```

**Rubi [A]** time = 0.09, antiderivative size = 83, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 19, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.158, Rules used = {3828, 3827, 136}

$$\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m F_1\left(m+\frac{1}{2};\frac{1}{2},2;m+\frac{3}{2};\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[e + f*x])^m*Sin[e + f*x],x]
```

```
[Out] (Sqrt[2]*AppellF1[1/2 + m, 1/2, 2, 3/2 + m, (1 + Csc[e + f*x])/2, 1 + Csc[e
 + f*x]]*Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + 2*m)*Sqrt[1 - Csc[e +
 f*x]])
```

Rule 136

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Simp[((b*e - a*f)^p*(a + b*x)^(m + 1)*AppellF1[m + 1, -
n, -p, m + 2, -((d*(a + b*x))/(b*c - a*d)), -((f*(a + b*x))/(b*e - a*f))])/
(b^(p + 1)*(m + 1)*(b/(b*c - a*d))^n), x] /; FreeQ[{a, b, c, d, e, f, m, n}
, x] && !IntegerQ[m] && !IntegerQ[n] && IntegerQ[p] && GtQ[b/(b*c - a*d),
 0] && !(GtQ[d/(d*a - c*b), 0] && SimplerQ[c + d*x, a + b*x])
```

Rule 3827

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*
x]]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[((d*x)^(n - 1)*(a + b*x)^(m - 1/2)
)/Sqrt[a - b*x], x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n}, x
] && EqQ[a^2 - b^2, 0] && !IntegerQ[m] && GtQ[a, 0]
```

Rule 3828

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^IntPart[m]*(a + b*Csc[e + f*x])^FracPart[m
])/(1 + (b*Csc[e + f*x])/a)^FracPart[m], Int[(1 + (b*Csc[e + f*x])/a)^m*(d*
Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^2
, 0] && !IntegerQ[m] && !GtQ[a, 0]
```

Rubi steps

$$\int (a + a\csc(e + fx))^m \sin(e + fx)\,dx = \left((1 + \csc(e + fx))^{-m}(a + a\csc(e + fx))^m\right) \int (1 + \csc(e + fx))^m \sin(e$$

$$= \frac{\left(\cot(e + fx)(1 + \csc(e + fx))^{-\frac{1}{2}-m}(a + a\csc(e + fx))^m\right) \text{Subst}\left(\int \frac{(1+x}{\sqrt{1}}\right.}{f\sqrt{1 - \csc(e + fx)}}$$

$$= \frac{\sqrt{2}F_1\left(\frac{1}{2} + m; \frac{1}{2}, 2; \frac{3}{2} + m; \frac{1}{2}(1 + \csc(e + fx)), 1 + \csc(e + fx)\right)\cot(e +}{f(1 + 2m)\sqrt{1 - \csc(e + fx)}}$$

**Mathematica [F]** time = 3.77, size = 0, normalized size = 0.00

$$\int (a + a\csc(e + fx))^m \sin(e + fx)\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x],x]
[Out] Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x], x]
```

**fricas [F]** time = 0.53, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(a\csc\left(fx + e\right) + a\right)^m \sin\left(fx + e\right), x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m*sin(f*x+e),x, algorithm="fricas")
[Out] integral((a*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx + e\right) + a\right)^m \sin\left(fx + e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m*sin(f*x+e),x, algorithm="giac")
[Out] integrate((a*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

**maple [F]** time = 2.26, size = 0, normalized size = 0.00

$$\int \left(a + a\csc\left(fx + e\right)\right)^m \sin\left(fx + e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(f*x+e))^m*sin(f*x+e),x)
[Out] int((a+a*csc(f*x+e))^m*sin(f*x+e),x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx + e\right) + a\right)^m \sin\left(fx + e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m*sin(f*x+e),x, algorithm="maxima")
```

```
[Out] integrate((a*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

**mupad [F]**   time = 0.00, size = -1, normalized size = -0.01

$$\int \sin\left(e+f\,x\right)\,{\left(a+\frac{a}{\sin\left(e+f\,x\right)}\right)}^m \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(e + f*x)*(a + a/sin(e + f*x))^m,x)
```

```
[Out] int(sin(e + f*x)*(a + a/sin(e + f*x))^m, x)
```

**sympy [F]**   time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a \left(\csc\left(e + f x\right) + 1\right)\right)^{m} \sin\left(e + f x\right)\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))**m*sin(f*x+e),x)
```

```
[Out] Integral((a*(csc(e + f*x) + 1))**m*sin(e + f*x), x)
```

# 3.35 $\int (a + a\csc(e + fx))^m \sin^2(e + fx)\,dx$

**Optimal**. Leaf size=84

$$-\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m F_1\left(m+\frac{1}{2};\frac{1}{2},3;m+\frac{3}{2};\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

```
[Out] -AppellF1(1/2+m,3,1/2,3/2+m,1+csc(f*x+e),1/2+1/2*csc(f*x+e))*cot(f*x+e)*(a+
a*csc(f*x+e))^m*2^(1/2)/f/(1+2*m)/(1-csc(f*x+e))^(1/2)
```

**Rubi [A]** time = 0.11, antiderivative size = 84, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 21, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.143, Rules used = {3828, 3827, 136}

$$-\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m F_1\left(m+\frac{1}{2};\frac{1}{2},3;m+\frac{3}{2};\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

Antiderivative was successfully verified.

```
[In] Int[(a + a*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
```

```
[Out] -((Sqrt[2]*AppellF1[1/2 + m, 1/2, 3, 3/2 + m, (1 + Csc[e + f*x])/2, 1 + Csc
[e + f*x]]*Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + 2*m)*Sqrt[1 - Csc[e
 + f*x]]))
```

Rule 136

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Simp[((b*e - a*f)^p*(a + b*x)^(m + 1)*AppellF1[m + 1, -
n, -p, m + 2, -((d*(a + b*x))/(b*c - a*d)), -((f*(a + b*x))/(b*e - a*f))])/
(b^(p + 1)*(m + 1)*(b/(b*c - a*d))^n), x] /; FreeQ[{a, b, c, d, e, f, m, n}
, x] &&  !IntegerQ[m] &&  !IntegerQ[n] && IntegerQ[p] && GtQ[b/(b*c - a*d),
 0] &&  !(GtQ[d/(d*a - c*b), 0] && SimplerQ[c + d*x, a + b*x])
```

Rule 3827

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^2*d*Cot[e + f*x])/(f*Sqrt[a + b*Csc[e + f*
x]]*Sqrt[a - b*Csc[e + f*x]]), Subst[Int[((d*x)^(n - 1)*(a + b*x)^(m - 1/2)
)/Sqrt[a - b*x], x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n}, x
] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

Rule 3828

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Dist[(a^IntPart[m]*(a + b*Csc[e + f*x])^FracPart[m
])/(1 + (b*Csc[e + f*x])/a)^FracPart[m], Int[(1 + (b*Csc[e + f*x])/a)^m*(d*
Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^2
, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

Rubi steps

$$\int (a+a\csc(e+fx))^m \sin^2(e+fx)\,dx = \left((1+\csc(e+fx))^{-m}(a+a\csc(e+fx))^m\right)\int (1+\csc(e+fx))^m \sin^2(e+$$

$$= \frac{\left(\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m\right)\text{Subst}\left(\int \frac{(1+x)^{-}}{\sqrt{1-x}}\right.}{f\sqrt{1-\csc(e+fx)}}$$

$$= -\frac{\sqrt{2}F_1\left(\frac{1}{2}+m;\frac{1}{2},3;\frac{3}{2}+m;\frac{1}{2}(1+\csc(e+fx)),1+\csc(e+fx)\right)\cot(e+}{f(1+2m)\sqrt{1-\csc(e+fx)}}$$

**Mathematica** [F] time = 7.71, size = 0, normalized size = 0.00

$$\int (a+a\csc(e+fx))^m \sin^2(e+fx)\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
[Out] Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x]^2, x]
```

**fricas** [F] time = 0.61, size = 0, normalized size = 0.00

$$\text{integral}\left(-\left(\cos\left(fx+e\right)^2-1\right)\left(a\csc\left(fx+e\right)+a\right)^m, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="fricas")
[Out] integral(-(cos(f*x + e)^2 - 1)*(a*csc(f*x + e) + a)^m, x)
```

**giac** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m \sin\left(fx+e\right)^2\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="giac")
[Out] integrate((a*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

**maple** [F] time = 4.69, size = 0, normalized size = 0.00

$$\int \left(a+a\csc\left(fx+e\right)\right)^m \left(\sin^2\left(fx+e\right)\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x)
[Out] int((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

**maxima** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a\csc\left(fx+e\right)+a\right)^m \sin\left(fx+e\right)^2\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="maxima")
```

```
[Out] integrate((a*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

**mupad** **[F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \sin\left(e+f\,x\right)^2\,{\left(a+\frac{a}{\sin\left(e+f\,x\right)}\right)}^m \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(e + f*x)^2*(a + a/sin(e + f*x))^m,x)
[Out] int(sin(e + f*x)^2*(a + a/sin(e + f*x))^m, x)
```

**sympy** **[F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a \left(\csc{\left(e + f x \right)} + 1\right)\right)^{m} \sin^{2}{\left(e + f x \right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(f*x+e))**m*sin(f*x+e)**2,x)
[Out] Integral((a*(csc(e + f*x) + 1))**m*sin(e + f*x)**2, x)
```

## 3.36 $\int (a+b\csc(c+dx))^4\,dx$

**Optimal**. Leaf size=107

$$a^4x-\frac{b^2\left(17a^2+2b^2\right)\cot(c+dx)}{3d}-\frac{2ab\left(2a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{d}-\frac{4ab^3\cot(c+dx)\csc(c+dx)}{3d}-\frac{b^2\cot(c+d}{}$$

```
[Out] a^4*x-2*a*b*(2*a^2+b^2)*arctanh(cos(d*x+c))/d-1/3*b^2*(17*a^2+2*b^2)*cot(d*
x+c)/d-4/3*a*b^3*cot(d*x+c)*csc(d*x+c)/d-1/3*b^2*cot(d*x+c)*(a+b*csc(d*x+c)
)^2/d
```

**Rubi [A]** time = 0.11, antiderivative size = 107, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.417, Rules used = {3782, 4048, 3770, 3767, 8}

$$-\frac{b^2\left(17a^2+2b^2\right)\cot(c+dx)}{3d}-\frac{2ab\left(2a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{d}+a^4x-\frac{4ab^3\cot(c+dx)\csc(c+dx)}{3d}-\frac{b^2\cot(c+}{}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^4,x]
```

```
[Out] a^4*x - (2*a*b*(2*a^2 + b^2)*ArcTanh[Cos[c + d*x]])/d - (b^2*(17*a^2 + 2*b^
2)*Cot[c + d*x])/(3*d) - (4*a*b^3*Cot[c + d*x]*Csc[c + d*x])/(3*d) - (b^2*C
ot[c + d*x]*(a + b*Csc[c + d*x])^2)/(3*d)
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 3767

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3782

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(b^2*Co
t[c + d*x]*(a + b*Csc[c + d*x])^(n - 2))/(d*(n - 1)), x] + Dist[1/(n - 1),
Int[(a + b*Csc[c + d*x])^(n - 3)*Simp[a^3*(n - 1) + (b*(b^2*(n - 2) + 3*a^2
*(n - 1)))*Csc[c + d*x] + (a*b^2*(3*n - 4))*Csc[c + d*x]^2, x], x], x] /; F
reeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[n, 2] && IntegerQ[2*n]
```

Rule 4048

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbol] :> -Simp[(b*C*Csc[e +
f*x]*Cot[e + f*x])/(2*f), x] + Dist[1/2, Int[Simp[2*A*a + (2*B*a + b*(2*A +
 C))*Csc[e + f*x] + 2*(a*C + B*b)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b
, e, f, A, B, C}, x]
```

Rubi steps

$$\int (a+b\csc(c+dx))^4\,dx = -\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}+\frac{1}{3}\int (a+b\csc(c+dx))\left(3a^3+b\left(9a^2+2b^2\right)\right.$$

$$= -\frac{4ab^3\cot(c+dx)\csc(c+dx)}{3d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}+\frac{1}{6}\int \left(6a^4+1\right.$$

$$= a^4x-\frac{4ab^3\cot(c+dx)\csc(c+dx)}{3d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}+\left(2ab\left(2a\right.\right.$$

$$= a^4x-\frac{2ab\left(2a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{d}-\frac{4ab^3\cot(c+dx)\csc(c+dx)}{3d}-\frac{b^2\cot(}{}$$

$$= a^4x-\frac{2ab\left(2a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{d}-\frac{b^2\left(17a^2+2b^2\right)\cot(c+dx)}{3d}-\frac{4ab^3\text{co}}{}$$

**Mathematica [B]** time = 6.25, size = 568, normalized size = 5.31

$$\frac{a^4(c+dx)\sin^4(c+dx)(a+b\csc(c+dx))^4}{d(a\sin(c+dx)+b)^4}+\frac{2\left(2a^3b+ab^3\right)\sin^4(c+dx)\log\left(\sin\left(\frac{1}{2}(c+dx)\right)\right)(a+b\csc(c+dx}{d(a\sin(c+dx)+b)^4}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^4,x]
```

```
[Out] (a^4*(c + d*x)*(a + b*Csc[c + d*x])^4*Sin[c + d*x]^4)/(d*(b + a*Sin[c + d*x
])^4) + ((-9*a^2*b^2*Cos[(c + d*x)/2] - b^4*Cos[(c + d*x)/2])*Csc[(c + d*x)
/2]*(a + b*Csc[c + d*x])^4*Sin[c + d*x]^4)/(3*d*(b + a*Sin[c + d*x])^4) - (
a*b^3*Csc[(c + d*x)/2]^2*(a + b*Csc[c + d*x])^4*Sin[c + d*x]^4)/(2*d*(b + a
*Sin[c + d*x])^4) - (b^4*Cot[(c + d*x)/2]*Csc[(c + d*x)/2]^2*(a + b*Csc[c +
 d*x])^4*Sin[c + d*x]^4)/(24*d*(b + a*Sin[c + d*x])^4) - (2*(2*a^3*b + a*b^
3)*(a + b*Csc[c + d*x])^4*Log[Cos[(c + d*x)/2]]*Sin[c + d*x]^4)/(d*(b + a*S
in[c + d*x])^4) + (2*(2*a^3*b + a*b^3)*(a + b*Csc[c + d*x])^4*Log[Sin[(c +
d*x)/2]]*Sin[c + d*x]^4)/(d*(b + a*Sin[c + d*x])^4) + (a*b^3*(a + b*Csc[c +
 d*x])^4*Sec[(c + d*x)/2]^2*Sin[c + d*x]^4)/(2*d*(b + a*Sin[c + d*x])^4) +
((a + b*Csc[c + d*x])^4*Sec[(c + d*x)/2]*(9*a^2*b^2*Sin[(c + d*x)/2] + b^4*
Sin[(c + d*x)/2])*Sin[c + d*x]^4)/(3*d*(b + a*Sin[c + d*x])^4) + (b^4*(a +
b*Csc[c + d*x])^4*Sec[(c + d*x)/2]^2*Sin[c + d*x]^4*Tan[(c + d*x)/2])/(24*d
*(b + a*Sin[c + d*x])^4)
```

**fricas [B]** time = 0.78, size = 217, normalized size = 2.03

$$-\frac{2\left(9\,a^2b^2+b^4\right)\cos\left(dx+c\right)^3-3\left(2\,a^3b+ab^3-\left(2\,a^3b+ab^3\right)\cos\left(dx+c\right)^2\right)\log\left(\frac{1}{2}\,\cos\left(dx+c\right)+\frac{1}{2}\right)\sin\left(dx+\right.}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^4,x, algorithm="fricas")
```

```
[Out] -1/3*(2*(9*a^2*b^2 + b^4)*cos(d*x + c)^3 - 3*(2*a^3*b + a*b^3 - (2*a^3*b +
a*b^3)*cos(d*x + c)^2)*log(1/2*cos(d*x + c) + 1/2)*sin(d*x + c) + 3*(2*a^3*
b + a*b^3 - (2*a^3*b + a*b^3)*cos(d*x + c)^2)*log(-1/2*cos(d*x + c) + 1/2)*
sin(d*x + c) - 3*(6*a^2*b^2 + b^4)*cos(d*x + c) - 3*(a^4*d*x*cos(d*x + c)^2
 - a^4*d*x + 2*a*b^3*cos(d*x + c))*sin(d*x + c))/((d*cos(d*x + c)^2 - d)*si
n(d*x + c))
```

**giac [B]** time = 0.78, size = 205, normalized size = 1.92

$$\frac{b^4\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^3+12\,ab^3\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2+24\,(dx+c)a^4+72\,a^2b^2\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+9\,b^4\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,\right.}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^4,x, algorithm="giac")
```

```
[Out] 1/24*(b^4*tan(1/2*d*x + 1/2*c)^3 + 12*a*b^3*tan(1/2*d*x + 1/2*c)^2 + 24*(d*
x + c)*a^4 + 72*a^2*b^2*tan(1/2*d*x + 1/2*c) + 9*b^4*tan(1/2*d*x + 1/2*c) +
 48*(2*a^3*b + a*b^3)*log(abs(tan(1/2*d*x + 1/2*c))) - (176*a^3*b*tan(1/2*d
*x + 1/2*c)^3 + 88*a*b^3*tan(1/2*d*x + 1/2*c)^3 + 72*a^2*b^2*tan(1/2*d*x +
1/2*c)^2 + 9*b^4*tan(1/2*d*x + 1/2*c)^2 + 12*a*b^3*tan(1/2*d*x + 1/2*c) + b
^4)/tan(1/2*d*x + 1/2*c)^3)/d
```

**maple [A]** time = 1.39, size = 139, normalized size = 1.30

$$a^4x+\frac{a^4c}{d}+\frac{4a^3b\ln(\csc(dx+c)-\cot(dx+c))}{d}-\frac{6a^2b^2\cot(dx+c)}{d}-\frac{2ab^3\cot(dx+c)\csc(dx+c)}{d}+\frac{2ab^3\ln(\csc(\ldots$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+b*csc(d*x+c))^4,x)
```

```
[Out] a^4*x+1/d*a^4*c+4/d*a^3*b*ln(csc(d*x+c)-cot(d*x+c))-6/d*a^2*b^2*cot(d*x+c)-
2*a*b^3*cot(d*x+c)*csc(d*x+c)/d+2/d*a*b^3*ln(csc(d*x+c)-cot(d*x+c))-2/3/d*b
^4*cot(d*x+c)-1/3/d*b^4*cot(d*x+c)*csc(d*x+c)^2
```

**maxima [A]** time = 0.34, size = 125, normalized size = 1.17

$$a^4x+\frac{ab^3\left(\frac{2\cos(dx+c)}{\cos(dx+c)^2-1}-\log(\cos(dx+c)+1)+\log(\cos(dx+c)-1)\right)}{d}-\frac{4a^3b\log(\cot(dx+c)+\csc(dx+c))}{d}-\frac{\ldots}{d\,\ldots}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^4,x, algorithm="maxima")
```

```
[Out] a^4*x + a*b^3*(2*cos(d*x + c)/(cos(d*x + c)^2 - 1) - log(cos(d*x + c) + 1)
+ log(cos(d*x + c) - 1))/d - 4*a^3*b*log(cot(d*x + c) + csc(d*x + c))/d - 6
*a^2*b^2/(d*tan(d*x + c)) - 1/3*(3*tan(d*x + c)^2 + 1)*b^4/(d*tan(d*x + c)^
3)
```

**mupad [B]** time = 0.62, size = 314, normalized size = 2.93

$$\frac{b^4\tan\left(\frac{c}{2}+\frac{dx}{2}\right)^3}{24d}-\frac{b^4\cot\left(\frac{c}{2}+\frac{dx}{2}\right)^3}{24d}-\frac{3b^4\cot\left(\frac{c}{2}+\frac{dx}{2}\right)}{8d}+\frac{3b^4\tan\left(\frac{c}{2}+\frac{dx}{2}\right)}{8d}+\frac{2a^4\,\mathrm{atan}\left(\frac{\cos\left(\frac{c}{2}+\frac{dx}{2}\right)a^3+4\sin\left(\frac{c}{2}+\frac{dx}{2}\right)a^2b+\ldots}{-\sin\left(\frac{c}{2}+\frac{dx}{2}\right)a^3+4\cos\left(\frac{c}{2}+\frac{dx}{2}\right)a^2b\ldots}\right)}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(c + d*x))^4,x)
```

```
[Out] (b^4*tan(c/2 + (d*x)/2)^3)/(24*d) - (b^4*cot(c/2 + (d*x)/2)^3)/(24*d) - (3*
b^4*cot(c/2 + (d*x)/2))/(8*d) + (3*b^4*tan(c/2 + (d*x)/2))/(8*d) + (2*a^4*a
tan((a^3*cos(c/2 + (d*x)/2) + 2*b^3*sin(c/2 + (d*x)/2) + 4*a^2*b*sin(c/2 +
(d*x)/2))/(2*b^3*cos(c/2 + (d*x)/2) - a^3*sin(c/2 + (d*x)/2) + 4*a^2*b*cos(
c/2 + (d*x)/2))))/d + (2*a*b^3*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/
d + (4*a^3*b*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/d - (3*a^2*b^2*cot
(c/2 + (d*x)/2))/d - (a*b^3*cot(c/2 + (d*x)/2)^2)/(2*d) + (3*a^2*b^2*tan(c/
2 + (d*x)/2))/d + (a*b^3*tan(c/2 + (d*x)/2)^2)/(2*d)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int(a+b\csc(c+dx))^4\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))**4,x)
```

```
[Out] Integral((a + b*csc(c + d*x))**4, x)
```

## 3.37 $\int (a+b\csc(c+dx))^3\,dx$

**Optimal**. Leaf size=73

$$a^3x-\frac{b\left(6a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{2d}-\frac{5ab^2\cot(c+dx)}{2d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}$$

```
[Out] a^3*x-1/2*b*(6*a^2+b^2)*arctanh(cos(d*x+c))/d-5/2*a*b^2*cot(d*x+c)/d-1/2*b^
2*cot(d*x+c)*(a+b*csc(d*x+c))/d
```

**Rubi [A]** time = 0.05, antiderivative size = 73, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.333, Rules used = {3782, 3770, 3767, 8}

$$-\frac{b\left(6a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{2d}+a^3x-\frac{5ab^2\cot(c+dx)}{2d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^3,x]
```

```
[Out] a^3*x - (b*(6*a^2 + b^2)*ArcTanh[Cos[c + d*x]])/(2*d) - (5*a*b^2*Cot[c + d*
x])/(2*d) - (b^2*Cot[c + d*x]*(a + b*Csc[c + d*x]))/(2*d)
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 3767

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3782

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> -Simp[(b^2*Co
t[c + d*x]*(a + b*Csc[c + d*x])^(n - 2))/(d*(n - 1)), x] + Dist[1/(n - 1),
Int[(a + b*Csc[c + d*x])^(n - 3)*Simp[a^3*(n - 1) + (b*(b^2*(n - 2) + 3*a^2
*(n - 1)))*Csc[c + d*x] + (a*b^2*(3*n - 4))*Csc[c + d*x]^2, x], x], x] /; F
reeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[n, 2] && IntegerQ[2*n]
```

Rubi steps

$$\begin{aligned}
\int (a+b\csc(c+dx))^3\,dx &= -\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}+\frac{1}{2}\int \left(2a^3+b\left(6a^2+b^2\right)\csc(c+dx)+5ab^2\csc^2\right.\\
&= a^3x-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}+\frac{1}{2}\left(5ab^2\right)\int \csc^2(c+dx)\,dx+\frac{1}{2}\left(b\left(6a^2+b^2\right.\right.\\
&= a^3x-\frac{b\left(6a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{2d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}-\frac{\left(5ab^2\right)\ldots}{}\\
&= a^3x-\frac{b\left(6a^2+b^2\right)\tanh^{-1}(\cos(c+dx))}{2d}-\frac{5ab^2\cot(c+dx)}{2d}-\frac{b^2\cot(c+dx)(a+b\csc}{2d}
\end{aligned}$$

**Mathematica [B]** time = 0.63, size = 152, normalized size = 2.08

$$\frac{8a^3c+8a^3dx+24a^2b\log\left(\sin\left(\frac{1}{2}(c+dx)\right)\right)-24a^2b\log\left(\cos\left(\frac{1}{2}(c+dx)\right)\right)+12ab^2\tan\left(\frac{1}{2}(c+dx)\right)-12ab^2\mathrm{co}}{8}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^3,x]
```

```
[Out] (8*a^3*c + 8*a^3*d*x - 12*a*b^2*Cot[(c + d*x)/2] - b^3*Csc[(c + d*x)/2]^2 -
 24*a^2*b*Log[Cos[(c + d*x)/2]] - 4*b^3*Log[Cos[(c + d*x)/2]] + 24*a^2*b*Lo
g[Sin[(c + d*x)/2]] + 4*b^3*Log[Sin[(c + d*x)/2]] + b^3*Sec[(c + d*x)/2]^2
+ 12*a*b^2*Tan[(c + d*x)/2])/(8*d)
```

**fricas [B]** time = 0.55, size = 155, normalized size = 2.12

$$\frac{4\,a^3dx\cos\left(dx+c\right)^2-4\,a^3dx+12\,ab^2\cos\left(dx+c\right)\sin\left(dx+c\right)+2\,b^3\cos\left(dx+c\right)+\left(6\,a^2b+b^3-\left(6\,a^2b+b^3\right)\right.}{4\left(d\cos\left(dx\right.\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^3,x, algorithm="fricas")
```

```
[Out] 1/4*(4*a^3*d*x*cos(d*x + c)^2 - 4*a^3*d*x + 12*a*b^2*cos(d*x + c)*sin(d*x +
 c) + 2*b^3*cos(d*x + c) + (6*a^2*b + b^3 - (6*a^2*b + b^3)*cos(d*x + c)^2)
*log(1/2*cos(d*x + c) + 1/2) - (6*a^2*b + b^3 - (6*a^2*b + b^3)*cos(d*x + c
)^2)*log(-1/2*cos(d*x + c) + 1/2))/(d*cos(d*x + c)^2 - d)
```

**giac [A]** time = 0.47, size = 134, normalized size = 1.84

$$\frac{b^3\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2+8\left(dx+c\right)a^3+12\,ab^2\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+4\left(6\,a^2b+b^3\right)\log\left(\left|\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)\right|\right)-\frac{36\,a^2b\tan}{}}{8\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^3,x, algorithm="giac")
```

```
[Out] 1/8*(b^3*tan(1/2*d*x + 1/2*c)^2 + 8*(d*x + c)*a^3 + 12*a*b^2*tan(1/2*d*x +
1/2*c) + 4*(6*a^2*b + b^3)*log(abs(tan(1/2*d*x + 1/2*c))) - (36*a^2*b*tan(1
/2*d*x + 1/2*c)^2 + 6*b^3*tan(1/2*d*x + 1/2*c)^2 + 12*a*b^2*tan(1/2*d*x + 1
/2*c) + b^3)/tan(1/2*d*x + 1/2*c)^2)/d
```

**maple [A]** time = 1.37, size = 99, normalized size = 1.36

$$a^3x+\frac{a^3c}{d}+\frac{3a^2b\ln\left(\csc\left(dx+c\right)-\cot\left(dx+c\right)\right)}{d}-\frac{3a\,b^2\cot\left(dx+c\right)}{d}-\frac{b^3\csc\left(dx+c\right)\cot\left(dx+c\right)}{2d}+\frac{b^3\ln\left(\csc\left(dx+\right.\right.}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+b*csc(d*x+c))^3,x)
```

```
[Out] a^3*x+1/d*a^3*c+3/d*a^2*b*ln(csc(d*x+c)-cot(d*x+c))-3*a*b^2*cot(d*x+c)/d-1/
2/d*b^3*csc(d*x+c)*cot(d*x+c)+1/2/d*b^3*ln(csc(d*x+c)-cot(d*x+c))
```

**maxima [A]** time = 0.34, size = 95, normalized size = 1.30

$$a^3x+\frac{b^3\left(\frac{2\cos(dx+c)}{\cos(dx+c)^2-1}-\log\left(\cos\left(dx+c\right)+1\right)+\log\left(\cos\left(dx+c\right)-1\right)\right)}{4\,d}-\frac{3\,a^2b\log\left(\cot\left(dx+c\right)+\csc\left(dx+c\right)\right)}{d}-$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^3,x, algorithm="maxima")
```

```
[Out] a^3*x + 1/4*b^3*(2*cos(d*x + c)/(cos(d*x + c)^2 - 1) - log(cos(d*x + c) + 1
) + log(cos(d*x + c) - 1))/d - 3*a^2*b*log(cot(d*x + c) + csc(d*x + c))/d -
 3*a*b^2/(d*tan(d*x + c))
```

**mupad [B]** time = 0.42, size = 234, normalized size = 3.21

$$\frac{b^3\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2}{8\,d}-\frac{b^3\cot\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2}{8\,d}+\frac{b^3\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)}\right)}{2\,d}+\frac{2\,a^3\,\mathrm{atan}\left(\frac{2\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^3+6\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^2\,b+\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,b^3}{-2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^3+6\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^2\,b+\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,b^3}\right)}{d}+\frac{3\,a}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(c + d*x))^3,x)
```

```
[Out] (b^3*tan(c/2 + (d*x)/2)^2)/(8*d) - (b^3*cot(c/2 + (d*x)/2)^2)/(8*d) + (b^3*
log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/(2*d) + (2*a^3*atan((2*a^3*cos(
c/2 + (d*x)/2) + b^3*sin(c/2 + (d*x)/2) + 6*a^2*b*sin(c/2 + (d*x)/2))/(b^3*
cos(c/2 + (d*x)/2) - 2*a^3*sin(c/2 + (d*x)/2) + 6*a^2*b*cos(c/2 + (d*x)/2))
))/d + (3*a^2*b*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/d - (3*a*b^2*co
t(c/2 + (d*x)/2))/(2*d) + (3*a*b^2*tan(c/2 + (d*x)/2))/(2*d)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int (a + b\csc(c + dx))^3\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))**3,x)
```

```
[Out] Integral((a + b*csc(c + d*x))**3, x)
```

## 3.38 $\int (a+b\csc(c+dx))^2\,dx$

**Optimal**. Leaf size=34

$$a^2x-\frac{2ab\tanh^{-1}(\cos(c+dx))}{d}-\frac{b^2\cot(c+dx)}{d}$$

```
[Out] a^2*x-2*a*b*arctanh(cos(d*x+c))/d-b^2*cot(d*x+c)/d
```

**Rubi [A]** time = 0.03, antiderivative size = 34, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.333, Rules used = {3773, 3770, 3767, 8}

$$a^2x-\frac{2ab\tanh^{-1}(\cos(c+dx))}{d}-\frac{b^2\cot(c+dx)}{d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^2,x]
```

```
[Out] a^2*x - (2*a*b*ArcTanh[Cos[c + d*x]])/d - (b^2*Cot[c + d*x])/d
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 3767

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> -Dist[d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3773

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^2, x_Symbol] :> Simp[a^2*x, x] +
 (Dist[2*a*b, Int[Csc[c + d*x], x], x] + Dist[b^2, Int[Csc[c + d*x]^2, x],
x]) /; FreeQ[{a, b, c, d}, x]
```

Rubi steps

$$\begin{aligned}\int (a+b\csc(c+dx))^2\,dx &= a^2x+(2ab)\int \csc(c+dx)\,dx+b^2\int \csc^2(c+dx)\,dx\\ &= a^2x-\frac{2ab\tanh^{-1}(\cos(c+dx))}{d}-\frac{b^2\operatorname{Subst}\left(\int 1\,dx,x,\cot(c+dx)\right)}{d}\\ &= a^2x-\frac{2ab\tanh^{-1}(\cos(c+dx))}{d}-\frac{b^2\cot(c+dx)}{d}\end{aligned}$$

**Mathematica [B]** time = 0.18, size = 76, normalized size = 2.24

$$\frac{2a\left(ac+adx+2b\log\left(\sin\left(\frac{1}{2}(c+dx)\right)\right)-2b\log\left(\cos\left(\frac{1}{2}(c+dx)\right)\right)\right)+b^2\tan\left(\frac{1}{2}(c+dx)\right)+b^2\left(-\cot\left(\frac{1}{2}(c+dx\right.\right.}{2d}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^2,x]
```

```
[Out] (-(b^2*Cot[(c + d*x)/2]) + 2*a*(a*c + a*d*x - 2*b*Log[Cos[(c + d*x)/2]] + 2
*b*Log[Sin[(c + d*x)/2]]) + b^2*Tan[(c + d*x)/2])/(2*d)
```

**fricas [B]** time = 0.68, size = 77, normalized size = 2.26

$$\frac{a^2 dx \sin(dx+c) - ab \log\left(\frac{1}{2}\cos(dx+c)+\frac{1}{2}\right)\sin(dx+c) + ab\log\left(-\frac{1}{2}\cos(dx+c)+\frac{1}{2}\right)\sin(dx+c) - b^2\cos(d}{d\sin(dx+c)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^2,x, algorithm="fricas")
```

```
[Out] (a^2*d*x*sin(d*x + c) - a*b*log(1/2*cos(d*x + c) + 1/2)*sin(d*x + c) + a*b*
log(-1/2*cos(d*x + c) + 1/2)*sin(d*x + c) - b^2*cos(d*x + c))/(d*sin(d*x +
c))
```

**giac [B]** time = 0.86, size = 74, normalized size = 2.18

$$\frac{2(dx+c)a^2 + 4ab\log\left(\left|\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)\right|\right) + b^2\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right) - \frac{4ab\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+b^2}{\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^2,x, algorithm="giac")
```

```
[Out] 1/2*(2*(d*x + c)*a^2 + 4*a*b*log(abs(tan(1/2*d*x + 1/2*c))) + b^2*tan(1/2*d
*x + 1/2*c) - (4*a*b*tan(1/2*d*x + 1/2*c) + b^2)/tan(1/2*d*x + 1/2*c))/d
```

**maple [A]** time = 0.83, size = 52, normalized size = 1.53

$$a^2x - \frac{b^2\cot(dx+c)}{d} + \frac{2ab\ln(\csc(dx+c)-\cot(dx+c))}{d} + \frac{a^2c}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+b*csc(d*x+c))^2,x)
```

```
[Out] a^2*x-1/d*b^2*cot(d*x+c)+2/d*a*b*ln(csc(d*x+c)-cot(d*x+c))+1/d*a^2*c
```

**maxima [A]** time = 0.36, size = 43, normalized size = 1.26

$$a^2x - \frac{2ab\log(\cot(dx+c)+\csc(dx+c))}{d} - \frac{b^2}{d\tan(dx+c)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))^2,x, algorithm="maxima")
```

```
[Out] a^2*x - 2*a*b*log(cot(d*x + c) + csc(d*x + c))/d - b^2/(d*tan(d*x + c))
```

**mupad [B]** time = 0.33, size = 105, normalized size = 3.09

$$\frac{2a^2\,\mathrm{atan}\left(\frac{a\cos\left(\frac{c}{2}+\frac{dx}{2}\right)+2b\sin\left(\frac{c}{2}+\frac{dx}{2}\right)}{2b\cos\left(\frac{c}{2}+\frac{dx}{2}\right)-a\sin\left(\frac{c}{2}+\frac{dx}{2}\right)}\right)}{d} - \frac{b^2\cot(c+dx)}{d} + \frac{2ab\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{dx}{2}\right)}{\cos\left(\frac{c}{2}+\frac{dx}{2}\right)}\right)}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(c + d*x))^2,x)
```

```
[Out] (2*a^2*atan((a*cos(c/2 + (d*x)/2) + 2*b*sin(c/2 + (d*x)/2))/(2*b*cos(c/2 +
(d*x)/2) - a*sin(c/2 + (d*x)/2))))/d - (b^2*cot(c + d*x))/d + (2*a*b*log(si
n(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/d
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int (a + b \csc (c + dx))^2 \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(d*x+c))**2,x)
```

```
[Out] Integral((a + b*csc(c + d*x))**2, x)
```

## 3.39 $\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=112

$$\frac{a\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^4}-\frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3}-\frac{2a^4\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^4\sqrt{a^2-b^2}}+\frac{a\cot(x)\csc(x)}{2b^2}-\frac{\cot(x)\csc^2(x)}{3b}$$

```
[Out] 1/2*a*(2*a^2+b^2)*arctanh(cos(x))/b^4-1/3*(3*a^2+2*b^2)*cot(x)/b^3+1/2*a*co
t(x)*csc(x)/b^2-1/3*cot(x)*csc(x)^2/b-2*a^4*arctanh((a+b*tan(1/2*x))/(a^2-b
^2)^(1/2))/b^4/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.41, antiderivative size = 112, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 9, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.692, Rules used = {3851, 4092, 4082, 3998, 3770, 3831, 2660, 618, 206}

$$-\frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3}+\frac{a\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^4}-\frac{2a^4\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^4\sqrt{a^2-b^2}}+\frac{a\cot(x)\csc(x)}{2b^2}-\frac{\cot(x)\csc^2(x)}{3b}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^5/(a + b*Csc[x]),x]
```

```
[Out] (a*(2*a^2 + b^2)*ArcTanh[Cos[x]])/(2*b^4) - (2*a^4*ArcTanh[(a + b*Tan[x/2])
/Sqrt[a^2 - b^2]])/(b^4*Sqrt[a^2 - b^2]) - ((3*a^2 + 2*b^2)*Cot[x])/(3*b^3)
 + (a*Cot[x]*Csc[x])/(2*b^2) - (Cot[x]*Csc[x]^2)/(3*b)
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3851

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> -Simp[(d^3*Cot[e + f*x]*(d*Csc[e + f*x])^(n - 3))/(b*f*(
n - 2)), x] + Dist[d^3/(b*(n - 2)), Int[((d*Csc[e + f*x])^(n - 3)*Simp[a*(n
 - 3) + b*(n - 3)*Csc[e + f*x] - a*(n - 2)*Csc[e + f*x]^2, x])/(a + b*Csc[e
 + f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && GtQ[n
, 3]
```

Rule 3998

```
Int[(csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(B_.) + (A_)))/(csc[(
e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbol] :> Dist[B/b, Int[Csc[e + f*x],
x], x] + Dist[(A*b - a*B)/b, Int[Csc[e + f*x]/(a + b*Csc[e + f*x]), x], x]
/; FreeQ[{a, b, e, f, A, B}, x] && NeQ[A*b - a*B, 0]
```

Rule 4082

```
Int[csc[(e_.) + (f_.)*(x_)]*((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e
_.) + (f_.)*(x_)]^2*(C_.))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_S
ymbol] :> -Simp[(C*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1))/(b*f*(m + 2))
, x] + Dist[1/(b*(m + 2)), Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*Simp[b*A
*(m + 2) + b*C*(m + 1) + (b*B*(m + 2) - a*C)*Csc[e + f*x], x], x], x] /; Fr
eeQ[{a, b, e, f, A, B, C, m}, x] &&  !LtQ[m, -1]
```

Rule 4092

```
Int[csc[(e_.) + (f_.)*(x_)]^2*((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[
(e_.) + (f_.)*(x_)]^2*(C_.))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x
_Symbol] :> -Simp[(C*Csc[e + f*x]*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)
)/(b*f*(m + 3)), x] + Dist[1/(b*(m + 3)), Int[Csc[e + f*x]*(a + b*Csc[e + f
*x])^m*Simp[a*C + b*(C*(m + 2) + A*(m + 3))*Csc[e + f*x] - (2*a*C - b*B*(m
+ 3))*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, e, f, A, B, C, m}, x] && N
eQ[a^2 - b^2, 0] &&  !LtQ[m, -1]
```

Rubi steps

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx = -\frac{\cot(x)\csc^2(x)}{3b} + \frac{\int \frac{\csc^2(x)\left(2a+2b\csc(x)-3a\csc^2(x)\right)}{a+b\csc(x)}dx}{3b}$$

$$= \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b} + \frac{\int \frac{\csc(x)\left(-3a^2+ab\csc(x)+2\left(3a^2+2b^2\right)\csc^2(x)\right)}{a+b\csc(x)}dx}{6b^2}$$

$$= -\frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b} + \frac{\int \frac{\csc(x)\left(-3a^2b-3a\left(2a^2+b^2\right)\csc(x)\right)}{a+b\csc(x)}dx}{6b^3}$$

$$= -\frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b} + \frac{a^4\int \frac{\csc(x)}{a+b\csc(x)}dx}{b^4} - \frac{\left(a\left(2a^2+b^2\right)\right)}{2b^4}$$

$$= \frac{a\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^4} - \frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b} + \frac{a^4\int}{}$$

$$= \frac{a\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^4} - \frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b} + \frac{\left(2a^4\right.}{}$$

$$= \frac{a\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^4} - \frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b} - \frac{\left(4a^4\right.}{}$$

$$= \frac{a\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^4} - \frac{2a^4\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{b^4\sqrt{a^2-b^2}} - \frac{\left(3a^2+2b^2\right)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(}{2b^2}$$

**Mathematica** [A] time = 1.73, size = 125, normalized size = 1.12

$$\frac{b\left(3a^2+2b^2\right)\cos(3x)\csc^3(x) - 3b\cot(x)\csc(x)\left(\left(a^2+2b^2\right)\csc(x)-2ab\right) + 6a\left(2a^2+b^2\right)\left(\log\left(\cos\left(\frac{x}{2}\right)\right) - \log\left(\mathrm{si}\right.\right.}{12b^4}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^5/(a + b*Csc[x]),x]
```

```
[Out] ((24*a^4*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2] + b*(3
*a^2 + 2*b^2)*Cos[3*x]*Csc[x]^3 - 3*b*Cot[x]*Csc[x]*(-2*a*b + (a^2 + 2*b^2)
*Csc[x]) + 6*a*(2*a^2 + b^2)*(Log[Cos[x/2]] - Log[Sin[x/2]]))/(12*b^4)
```

**fricas** [B] time = 0.74, size = 607, normalized size = 5.42

$$\left[\frac{4\left(3a^4b-a^2b^3-2b^5\right)\cos(x)^3 - 6\left(a^4\cos(x)^2-a^4\right)\sqrt{a^2-b^2}\,\log\left(-\frac{\left(a^2-2b^2\right)\cos(x)^2+2ab\sin(x)+a^2+b^2-2\left(b\cos(x)\sin(x)+a\right.}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right.}{}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^5/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/12*(4*(3*a^4*b - a^2*b^3 - 2*b^5)*cos(x)^3 - 6*(a^4*cos(x)^2 - a^4)*sqrt
(a^2 - b^2)*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 + b^2 - 2*(b*
cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b*sin(x) - a
^2 - b^2))*sin(x) + 6*(a^3*b^2 - a*b^4)*cos(x)*sin(x) + 3*(2*a^5 - a^3*b^2
```

```
- a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*cos(x)^2)*log(1/2*cos(x) + 1/2)*sin(x)
- 3*(2*a^5 - a^3*b^2 - a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*cos(x)^2)*log(-1/2
*cos(x) + 1/2)*sin(x) - 12*(a^4*b - b^5)*cos(x))/((a^2*b^4 - b^6 - (a^2*b^4
 - b^6)*cos(x)^2)*sin(x)), 1/12*(4*(3*a^4*b - a^2*b^3 - 2*b^5)*cos(x)^3 + 1
2*(a^4*cos(x)^2 - a^4)*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(x)
+ a)/((a^2 - b^2)*cos(x)))*sin(x) + 6*(a^3*b^2 - a*b^4)*cos(x)*sin(x) + 3*(
2*a^5 - a^3*b^2 - a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*cos(x)^2)*log(1/2*cos(x
) + 1/2)*sin(x) - 3*(2*a^5 - a^3*b^2 - a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*co
s(x)^2)*log(-1/2*cos(x) + 1/2)*sin(x) - 12*(a^4*b - b^5)*cos(x))/((a^2*b^4
- b^6 - (a^2*b^4 - b^6)*cos(x)^2)*sin(x))]
```

**giac [A]** time = 0.66, size = 194, normalized size = 1.73

$$\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a^4}{\sqrt{-a^2+b^2}\,b^4}+\frac{b^2\tan\left(\frac{1}{2}x\right)^3-3\,ab\tan\left(\frac{1}{2}x\right)^2+12\,a^2\tan\left(\frac{1}{2}x\right)+9\,b^2\tan\left(\frac{1}{2}\right.}{24\,b^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^5/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] 2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
b^2)))*a^4/(sqrt(-a^2 + b^2)*b^4) + 1/24*(b^2*tan(1/2*x)^3 - 3*a*b*tan(1/2*
x)^2 + 12*a^2*tan(1/2*x) + 9*b^2*tan(1/2*x))/b^3 - 1/2*(2*a^3 + a*b^2)*log(
abs(tan(1/2*x)))/b^4 + 1/24*(44*a^3*tan(1/2*x)^3 + 22*a*b^2*tan(1/2*x)^3 -
12*a^2*b*tan(1/2*x)^2 - 9*b^3*tan(1/2*x)^2 + 3*a*b^2*tan(1/2*x) - b^3)/(b^4
*tan(1/2*x)^3)
```

**maple [A]** time = 0.27, size = 162, normalized size = 1.45

$$\frac{\tan^3\left(\frac{x}{2}\right)}{24b}-\frac{a\left(\tan^2\left(\frac{x}{2}\right)\right)}{8b^2}+\frac{a^2\tan\left(\frac{x}{2}\right)}{2b^3}+\frac{3\tan\left(\frac{x}{2}\right)}{8b}+\frac{2a^4\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{b^4\sqrt{-a^2+b^2}}-\frac{1}{24b\tan\left(\frac{x}{2}\right)^3}-\frac{a^2}{2b^3\tan\left(\frac{x}{2}\right)}-\frac{3}{8b\tan\left(\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^5/(a+b*csc(x)),x)
```

```
[Out] 1/24/b*tan(1/2*x)^3-1/8/b^2*a*tan(1/2*x)^2+1/2/b^3*a^2*tan(1/2*x)+3/8/b*tan
(1/2*x)+2/b^4*a^4/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^
2)^(1/2))-1/24/b/tan(1/2*x)^3-1/2/b^3/tan(1/2*x)*a^2-3/8/b/tan(1/2*x)+1/8*a
/b^2/tan(1/2*x)^2-1/b^4*a^3*ln(tan(1/2*x))-1/2/b^2*a*ln(tan(1/2*x))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^5/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.79, size = 588, normalized size = 5.25

$$-\frac{b^2\left(\frac{3\,a\,\sin(2\,x)\,\sqrt{a^2-b^2}}{4}+\frac{3\,a\,\sin(3\,x)\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{8}-\frac{9\,a\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sin(x)\,\sqrt{a^2-b^2}}{8}\right)+b^3\,\left(\frac{\cos(3\,x)\,\sqrt{a^2-b^2}}{2}-\frac{3\,\cos(x)\,\sqrt{a^2-b^2}}{2}\right)\cdots}{\cdots}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^5*(a + b/sin(x))),x)
```

```
[Out] -(b^2*((3*a*sin(2*x)*(a^2 - b^2)^(1/2))/4 + (3*a*sin(3*x)*log(sin(x/2)/cos(
x/2))*(a^2 - b^2)^(1/2))/8 - (9*a*log(sin(x/2)/cos(x/2))*sin(x)*(a^2 - b^2)
^(1/2))/8) + b^3*((cos(3*x)*(a^2 - b^2)^(1/2))/2 - (3*cos(x)*(a^2 - b^2)^(1
/2))/2) - b*((3*a^2*cos(x)*(a^2 - b^2)^(1/2))/4 - (3*a^2*cos(3*x)*(a^2 - b^
2)^(1/2))/4) + (a^4*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*sin(x/2)*
(a^2 - b^2)^(1/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b*cos(x/2)
*(a^2 - b^2)^(1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*cos(x/2) +
4*a^3*b^2*sin(x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*sin(x)*9i)/2 - (
a^4*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*sin(x/2)*(a^2 - b^2)^(1/2
)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b*cos(x/2)*(a^2 - b^2)^(1/
2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*cos(x/2) + 4*a^3*b^2*sin(x/
2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*sin(3*x)*3i)/2 - (9*a^3*log(sin(
x/2)/cos(x/2))*sin(x)*(a^2 - b^2)^(1/2))/4 + (3*a^3*sin(3*x)*log(sin(x/2)/c
os(x/2))*(a^2 - b^2)^(1/2))/4)/((3*b^4*sin(3*x)*(a^2 - b^2)^(1/2))/4 - (9*b
^4*sin(x)*(a^2 - b^2)^(1/2))/4)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**5/(a+b*csc(x)),x)
```

```
[Out] Integral(csc(x)**5/(a + b*csc(x)), x)
```

## 3.40 $\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=84

$$-\frac{(2a^2+b^2)\tanh^{-1}(\cos(x))}{2b^3}+\frac{2a^3\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^3\sqrt{a^2-b^2}}+\frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}$$

```
[Out] -1/2*(2*a^2+b^2)*arctanh(cos(x))/b^3+a*cot(x)/b^2-1/2*cot(x)*csc(x)/b+2*a^3
*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/b^3/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.25, antiderivative size = 84, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 8, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.615, Rules used = {3851, 4082, 3998, 3770, 3831, 2660, 618, 206}

$$-\frac{(2a^2+b^2)\tanh^{-1}(\cos(x))}{2b^3}+\frac{2a^3\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^3\sqrt{a^2-b^2}}+\frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^4/(a + b*Csc[x]),x]
```

```
[Out] -((2*a^2 + b^2)*ArcTanh[Cos[x]])/(2*b^3) + (2*a^3*ArcTanh[(a + b*Tan[x/2])/
Sqrt[a^2 - b^2]])/(b^3*Sqrt[a^2 - b^2]) + (a*Cot[x])/b^2 - (Cot[x]*Csc[x])/
(2*b)
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3851

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> -Simp[(d^3*Cot[e + f*x]*(d*Csc[e + f*x])^(n - 3))/(b*f*(
n - 2)), x] + Dist[d^3/(b*(n - 2)), Int[((d*Csc[e + f*x])^(n - 3)*Simp[a*(n
 - 3) + b*(n - 3)*Csc[e + f*x] - a*(n - 2)*Csc[e + f*x]^2, x])/(a + b*Csc[e
 + f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && GtQ[n
, 3]
```

Rule 3998

```
Int[(csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(B_.) + (A_)))/(csc[(
e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbol] :> Dist[B/b, Int[Csc[e + f*x],
x], x] + Dist[(A*b - a*B)/b, Int[Csc[e + f*x]/(a + b*Csc[e + f*x]), x], x]
/; FreeQ[{a, b, e, f, A, B}, x] && NeQ[A*b - a*B, 0]
```

Rule 4082

```
Int[csc[(e_.) + (f_.)*(x_)]*((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e
_.) + (f_.)*(x_)]^2*(C_.))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_S
ymbol] :> -Simp[(C*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1))/(b*f*(m + 2))
, x] + Dist[1/(b*(m + 2)), Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*Simp[b*A
*(m + 2) + b*C*(m + 1) + (b*B*(m + 2) - a*C)*Csc[e + f*x], x], x], x] /; Fr
eeQ[{a, b, e, f, A, B, C, m}, x] && !LtQ[m, -1]
```

Rubi steps

$$\begin{aligned}
\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx &= -\frac{\cot(x)\csc(x)}{2b}+\frac{\int \frac{\csc(x)\left(a+b\csc(x)-2a\csc^2(x)\right)}{a+b\csc(x)}\,dx}{2b}\\
&= \frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}+\frac{\int \frac{\csc(x)\left(ab+\left(2a^2+b^2\right)\csc(x)\right)}{a+b\csc(x)}\,dx}{2b^2}\\
&= \frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}-\frac{a^3\int \frac{\csc(x)}{a+b\csc(x)}\,dx}{b^3}+\frac{\left(2a^2+b^2\right)\int \csc(x)\,dx}{2b^3}\\
&= -\frac{\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^3}+\frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}-\frac{a^3\int \frac{1}{1+\frac{a\sin(x)}{b}}\,dx}{b^4}\\
&= -\frac{\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^3}+\frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}-\frac{\left(2a^3\right)\text{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx,x,\tan\right.}{b^4}\\
&= -\frac{\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^3}+\frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}+\frac{\left(4a^3\right)\text{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx,x,\frac{2}{b}\right.}{b^4}\\
&= -\frac{\left(2a^2+b^2\right)\tanh^{-1}(\cos(x))}{2b^3}+\frac{2a^3\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{b^3\sqrt{a^2-b^2}}+\frac{a\cot(x)}{b^2}-\frac{\cot(x)\csc(x)}{2b}
\end{aligned}$$

**Mathematica [A]** time = 0.50, size = 144, normalized size = 1.71

$$\frac{8a^2\log\left(\sin\left(\frac{x}{2}\right)\right)-8a^2\log\left(\cos\left(\frac{x}{2}\right)\right)-\frac{16a^3\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)}{\sqrt{b^2-a^2}}-4ab\tan\left(\frac{x}{2}\right)+4ab\cot\left(\frac{x}{2}\right)-b^2\csc^2\left(\frac{x}{2}\right)+b^2\sec^2\left(\frac{x}{2}\right)}{8b^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^4/(a + b*Csc[x]),x]
```

```
[Out] ((-16*a^3*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2] + 4*a
*b*Cot[x/2] - b^2*Csc[x/2]^2 - 8*a^2*Log[Cos[x/2]] - 4*b^2*Log[Cos[x/2]] +
8*a^2*Log[Sin[x/2]] + 4*b^2*Log[Sin[x/2]] + b^2*Sec[x/2]^2 - 4*a*b*Tan[x/2]
)/(8*b^3)
```

**fricas [B]** time = 0.87, size = 524, normalized size = 6.24

$$\left[\frac{4\left(a^3b-ab^3\right)\cos(x)\sin(x)-2\left(a^3\cos(x)^2-a^3\right)\sqrt{a^2-b^2}\,\log\left(\frac{\left(a^2-2\,b^2\right)\cos(x)^2+2\,ab\sin(x)+a^2+b^2+2\,(b\cos(x)\sin(x)+a\cos}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right.}{}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^4/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/4*(4*(a^3*b - a*b^3)*cos(x)*sin(x) - 2*(a^3*cos(x)^2 - a^3)*sqrt(a^2 - b
^2)*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 + b^2 + 2*(b*cos(x)*si
n(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b*sin(x) - a^2 - b^2)
) - 2*(a^2*b^2 - b^4)*cos(x) - (2*a^4 - a^2*b^2 - b^4 - (2*a^4 - a^2*b^2 -
b^4)*cos(x)^2)*log(1/2*cos(x) + 1/2) + (2*a^4 - a^2*b^2 - b^4 - (2*a^4 - a^
2*b^2 - b^4)*cos(x)^2)*log(-1/2*cos(x) + 1/2))/(a^2*b^3 - b^5 - (a^2*b^3 -
b^5)*cos(x)^2), 1/4*(4*(a^3*b - a*b^3)*cos(x)*sin(x) - 4*(a^3*cos(x)^2 - a^
3)*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) + a)/((a^2 - b^2)*co
s(x))) - 2*(a^2*b^2 - b^4)*cos(x) - (2*a^4 - a^2*b^2 - b^4 - (2*a^4 - a^2*b
^2 - b^4)*cos(x)^2)*log(1/2*cos(x) + 1/2) + (2*a^4 - a^2*b^2 - b^4 - (2*a^4
 - a^2*b^2 - b^4)*cos(x)^2)*log(-1/2*cos(x) + 1/2))/(a^2*b^3 - b^5 - (a^2*b
^3 - b^5)*cos(x)^2)]
```

**giac [A]** time = 0.53, size = 141, normalized size = 1.68

$$-\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}\,x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a^3}{\sqrt{-a^2+b^2}\,b^3}+\frac{b\tan\left(\frac{1}{2}\,x\right)^2-4\,a\tan\left(\frac{1}{2}\,x\right)}{8\,b^2}+\frac{\left(2\,a^2+b^2\right)\log\left(\left|\tan\left(\frac{1}{2}\,x\right)\right|\right)}{2\,b^3}-\cdots$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^4/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] -2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*a^3/(sqrt(-a^2 + b^2)*b^3) + 1/8*(b*tan(1/2*x)^2 - 4*a*tan(1/2*x))/
b^2 + 1/2*(2*a^2 + b^2)*log(abs(tan(1/2*x)))/b^3 - 1/8*(12*a^2*tan(1/2*x)^2
 + 6*b^2*tan(1/2*x)^2 - 4*a*b*tan(1/2*x) + b^2)/(b^3*tan(1/2*x)^2)
```

**maple [A]** time = 0.23, size = 112, normalized size = 1.33

$$\frac{\tan^2\left(\frac{x}{2}\right)}{8b}-\frac{a\tan\left(\frac{x}{2}\right)}{2b^2}-\frac{2a^3\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{b^3\sqrt{-a^2+b^2}}-\frac{1}{8b\tan\left(\frac{x}{2}\right)^2}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)a^2}{b^3}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2b}+\frac{a}{2b^2\tan\left(\frac{x}{2}\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^4/(a+b*csc(x)),x)
```

```
[Out] 1/8/b*tan(1/2*x)^2-1/2/b^2*a*tan(1/2*x)-2/b^3*a^3/(-a^2+b^2)^(1/2)*arctan(1
/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2))-1/8/b/tan(1/2*x)^2+1/b^3*ln(tan(1
/2*x))*a^2+1/2/b*ln(tan(1/2*x))+1/2*a/b^2/tan(1/2*x)
```

**maxima** [F(-2)]   time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^4/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad** [B]   time = 0.58, size = 515, normalized size = 6.13

$$-\frac{b^2\left(\frac{\cos(x)\,\sqrt{a^2-b^2}}{2}-\frac{\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{4}+\frac{\cos(2\,x)\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{4}\right)-\frac{a^2\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{2}-\frac{a\,b\,\sin(2\,x)\,\sqrt{a^2-b^2}}{2}+\frac{a^2\,\cos(2\,x)\,\ln}{}}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^4*(a + b/sin(x))),x)
```

```
[Out] -(a^3*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*sin(x/2)*(a^2 - b^2)^(1
/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b*cos(x/2)*(a^2 - b^2)^(
1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*cos(x/2) + 4*a^3*b^2*sin(
x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*1i + b^2*((cos(x)*(a^2 - b^2)^
(1/2))/2 - (log(sin(x/2)/cos(x/2))*(a^2 - b^2)^(1/2))/4 + (cos(2*x)*log(sin
(x/2)/cos(x/2))*(a^2 - b^2)^(1/2))/4) - (a^2*log(sin(x/2)/cos(x/2))*(a^2 -
b^2)^(1/2))/2 - a^3*cos(2*x)*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*
sin(x/2)*(a^2 - b^2)^(1/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b
*cos(x/2)*(a^2 - b^2)^(1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*co
s(x/2) + 4*a^3*b^2*sin(x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*1i - (a
*b*sin(2*x)*(a^2 - b^2)^(1/2))/2 + (a^2*cos(2*x)*log(sin(x/2)/cos(x/2))*(a^
2 - b^2)^(1/2))/2)/((b^3*(a^2 - b^2)^(1/2))/2 - (b^3*cos(2*x)*(a^2 - b^2)^(
1/2))/2)
```

**sympy** [F]   time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**4/(a+b*csc(x)),x)
```

```
[Out] Integral(csc(x)**4/(a + b*csc(x)), x)
```

## 3.41 $\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=62

$$-\frac{2a^2\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^2\sqrt{a^2-b^2}}+\frac{a\tanh^{-1}(\cos(x))}{b^2}-\frac{\cot(x)}{b}$$

```
[Out] a*arctanh(cos(x))/b^2-cot(x)/b-2*a^2*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/
2))/b^2/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.15, antiderivative size = 62, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 7, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.538, Rules used = {3790, 3789, 3770, 3831, 2660, 618, 206}

$$-\frac{2a^2\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^2\sqrt{a^2-b^2}}+\frac{a\tanh^{-1}(\cos(x))}{b^2}-\frac{\cot(x)}{b}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^3/(a + b*Csc[x]),x]
```

```
[Out] (a*ArcTanh[Cos[x]])/b^2 - (2*a^2*ArcTanh[(a + b*Tan[x/2])/Sqrt[a^2 - b^2]])
/(b^2*Sqrt[a^2 - b^2]) - Cot[x]/b
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3789

```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Dist[1/b, Int[Csc[e + f*x], x], x] - Dist[a/b, Int[Csc[e + f*x]/(a
+ b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

Rule 3790

```
Int[csc[(e_.) + (f_.)*(x_)]^3/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> -Simp[Cot[e + f*x]/(b*f), x] - Dist[a/b, Int[Csc[e + f*x]^2/(a + b*
```

```
Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$
\begin{aligned}
\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx &= -\frac{\cot(x)}{b}-\frac{a\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx}{b}\\
&= -\frac{\cot(x)}{b}-\frac{a\int \csc(x)\,dx}{b^2}+\frac{a^2\int \frac{\csc(x)}{a+b\csc(x)}\,dx}{b^2}\\
&= \frac{a\tanh^{-1}(\cos(x))}{b^2}-\frac{\cot(x)}{b}+\frac{a^2\int \frac{1}{1+\frac{a\sin(x)}{b}}\,dx}{b^3}\\
&= \frac{a\tanh^{-1}(\cos(x))}{b^2}-\frac{\cot(x)}{b}+\frac{\left(2a^2\right)\mathrm{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{b^3}\\
&= \frac{a\tanh^{-1}(\cos(x))}{b^2}-\frac{\cot(x)}{b}-\frac{\left(4a^2\right)\mathrm{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx,x,\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b^3}\\
&= \frac{a\tanh^{-1}(\cos(x))}{b^2}-\frac{2a^2\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{b^2\sqrt{a^2-b^2}}-\frac{\cot(x)}{b}
\end{aligned}
$$

**Mathematica [A]** time = 0.20, size = 106, normalized size = 1.71

$$
\frac{\csc\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right)\left(2a^2\sin(x)\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)+\sqrt{b^2-a^2}\left(a\sin(x)\left(\log\left(\cos\left(\frac{x}{2}\right)\right)-\log\left(\sin\left(\frac{x}{2}\right)\right)\right)-b\cos(x)\right)\right)}{2b^2\sqrt{b^2-a^2}}
$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^3/(a + b*Csc[x]),x]
```

```
[Out] (Csc[x/2]*Sec[x/2]*(2*a^2*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]]*Sin[x]
+ Sqrt[-a^2 + b^2]*(-(b*Cos[x]) + a*(Log[Cos[x/2]] - Log[Sin[x/2]])*Sin[x])
))/(2*b^2*Sqrt[-a^2 + b^2])
```

**fricas [B]** time = 0.67, size = 308, normalized size = 4.97

$$
\left[\frac{\sqrt{a^2-b^2}\,a^2\log\left(-\frac{\left(a^2-2\,b^2\right)\cos(x)^2+2\,ab\sin(x)+a^2+b^2-2\,(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right)\sin(x)+\left(a^3-ab^2\right)\log\left(\frac{1}{2}\cos(x)+\right.}{2\left(a^2b^2-b^4\right)\sin(x)}\right.
$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^3/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/2*(sqrt(a^2 - b^2)*a^2*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2
 + b^2 - 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*
```

```
a*b*sin(x) - a^2 - b^2))*sin(x) + (a^3 - a*b^2)*log(1/2*cos(x) + 1/2)*sin(x
) - (a^3 - a*b^2)*log(-1/2*cos(x) + 1/2)*sin(x) - 2*(a^2*b - b^3)*cos(x))/(
(a^2*b^2 - b^4)*sin(x)), -1/2*(2*sqrt(-a^2 + b^2)*a^2*arctan(-sqrt(-a^2 + b
^2)*(b*sin(x) + a)/((a^2 - b^2)*cos(x)))*sin(x) - (a^3 - a*b^2)*log(1/2*cos
(x) + 1/2)*sin(x) + (a^3 - a*b^2)*log(-1/2*cos(x) + 1/2)*sin(x) + 2*(a^2*b
- b^3)*cos(x))/((a^2*b^2 - b^4)*sin(x))]
```

**giac [A]** time = 0.35, size = 98, normalized size = 1.58

$$\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a^2}{\sqrt{-a^2+b^2}\,b^2}-\frac{a\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{b^2}+\frac{\tan\left(\frac{1}{2}x\right)}{2b}+\frac{2a\tan\left(\frac{1}{2}x\right)-b}{2b^2\tan\left(\frac{1}{2}x\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^3/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] 2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
b^2)))*a^2/(sqrt(-a^2 + b^2)*b^2) - a*log(abs(tan(1/2*x)))/b^2 + 1/2*tan(1/
2*x)/b + 1/2*(2*a*tan(1/2*x) - b)/(b^2*tan(1/2*x))
```

**maple [A]** time = 0.21, size = 77, normalized size = 1.24

$$\frac{\tan\left(\frac{x}{2}\right)}{2b}+\frac{2a^2\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{b^2\sqrt{-a^2+b^2}}-\frac{1}{2b\tan\left(\frac{x}{2}\right)}-\frac{a\ln\left(\tan\left(\frac{x}{2}\right)\right)}{b^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^3/(a+b*csc(x)),x)
```

```
[Out] 1/2/b*tan(1/2*x)+2/b^2*a^2/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)
/(-a^2+b^2)^(1/2))-1/2/b/tan(1/2*x)-1/b^2*a*ln(tan(1/2*x))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^3/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.49, size = 135, normalized size = 2.18

$$-\frac{1}{b\tan(x)}-\frac{a\ln\left(\tan\left(\frac{x}{2}\right)\right)}{b^2}-\frac{a^2\,\mathrm{atan}\left(\frac{a^2\tan\left(\frac{x}{2}\right)\sqrt{a^2-b^2}\,4\mathrm{i}-b^2\tan\left(\frac{x}{2}\right)\sqrt{a^2-b^2}\,1\mathrm{i}+a\,b\,\sqrt{a^2-b^2}\,2\mathrm{i}}{4\tan\left(\frac{x}{2}\right)a^3+2a^2b-3\tan\left(\frac{x}{2}\right)a\,b^2-b^3}\right)2\mathrm{i}}{b^2\sqrt{a^2-b^2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^3*(a + b/sin(x))),x)
```

```
[Out] - 1/(b*tan(x)) - (a*log(tan(x/2)))/b^2 - (a^2*atan((a^2*tan(x/2)*(a^2 - b^2
)^(1/2)*4i - b^2*tan(x/2)*(a^2 - b^2)^(1/2)*1i + a*b*(a^2 - b^2)^(1/2)*2i)/
(4*a^3*tan(x/2) + 2*a^2*b - b^3 - 3*a*b^2*tan(x/2)))*2i)/(b^2*(a^2 - b^2)^(
1/2))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\csc^3(x)}{a + b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**3/(a+b*csc(x)),x)
```

```
[Out] Integral(csc(x)**3/(a + b*csc(x)), x)
```

## 3.42 $\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=53

$$\frac{2a\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}-\frac{\tanh^{-1}(\cos(x))}{b}$$

```
[Out] -arctanh(cos(x))/b+2*a*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/b/(a^2-b^2
)^(1/2)
```

**Rubi [A]** time = 0.11, antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.462, Rules used = {3789, 3770, 3831, 2660, 618, 206}

$$\frac{2a\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}-\frac{\tanh^{-1}(\cos(x))}{b}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]^2/(a + b*Csc[x]),x]
```

```
[Out] -(ArcTanh[Cos[x]]/b) + (2*a*ArcTanh[(a + b*Tan[x/2])/Sqrt[a^2 - b^2]])/(b*S
qrt[a^2 - b^2])
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rule 3789

```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Dist[1/b, Int[Csc[e + f*x], x], x] - Dist[a/b, Int[Csc[e + f*x]/(a
+ b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
```

```
}, x] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$
\begin{aligned}
\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx &= \frac{\int \csc(x)\,dx}{b}-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}\,dx}{b}\\
&= -\frac{\tanh^{-1}(\cos(x))}{b}-\frac{a\int \frac{1}{1+\frac{a\sin(x)}{b}}\,dx}{b^2}\\
&= -\frac{\tanh^{-1}(\cos(x))}{b}-\frac{(2a)\,\mathrm{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{b^2}\\
&= -\frac{\tanh^{-1}(\cos(x))}{b}+\frac{(4a)\,\mathrm{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx,x,\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b^2}\\
&= -\frac{\tanh^{-1}(\cos(x))}{b}+\frac{2a\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}
\end{aligned}
$$

**Mathematica [A]** time = 0.06, size = 62, normalized size = 1.17

$$
\frac{-\frac{2a\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)}{\sqrt{b^2-a^2}}+\log\left(\sin\left(\frac{x}{2}\right)\right)-\log\left(\cos\left(\frac{x}{2}\right)\right)}{b}
$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]^2/(a + b*Csc[x]),x]
```

```
[Out] ((-2*a*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2] - Log[Co
s[x/2]] + Log[Sin[x/2]])/b
```

**fricas [B]** time = 0.66, size = 245, normalized size = 4.62

$$
\left[\frac{\sqrt{a^2-b^2}\,a\log\left(\frac{\left(a^2-2\,b^2\right)\cos(x)^2+2\,ab\sin(x)+a^2+b^2+2\,(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right)-\left(a^2-b^2\right)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)+\left(a^2-\right.}{2\left(a^2b-b^3\right)}\right.
$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^2/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/2*(sqrt(a^2 - b^2)*a*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 +
b^2 + 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b
*sin(x) - a^2 - b^2)) - (a^2 - b^2)*log(1/2*cos(x) + 1/2) + (a^2 - b^2)*log
(-1/2*cos(x) + 1/2))/(a^2*b - b^3), 1/2*(2*sqrt(-a^2 + b^2)*a*arctan(-sqrt(
-a^2 + b^2)*(b*sin(x) + a)/((a^2 - b^2)*cos(x))) - (a^2 - b^2)*log(1/2*cos(
x) + 1/2) + (a^2 - b^2)*log(-1/2*cos(x) + 1/2))/(a^2*b - b^3)]
```

**giac [A]** time = 0.53, size = 63, normalized size = 1.19

$$
-\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a}{\sqrt{-a^2+b^2}\,b}+\frac{\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{b}
$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^2/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] -2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*a/(sqrt(-a^2 + b^2)*b) + log(abs(tan(1/2*x)))/b
```

**maple [A]** time = 0.22, size = 53, normalized size = 1.00

$$-\frac{2a\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{b\sqrt{-a^2+b^2}}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{b}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)^2/(a+b*csc(x)),x)
```

```
[Out] -2/b*a/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2))+1
/b*ln(tan(1/2*x))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)^2/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.43, size = 129, normalized size = 2.43

$$\frac{\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)}{b}-\frac{2\,a\,\mathrm{atanh}\left(\frac{\sqrt{a^2-b^2}\,\left(4\mathrm{i}\,\sin\left(\frac{x}{2}\right)\,a^2+2\mathrm{i}\,\cos\left(\frac{x}{2}\right)\,a\,b-1\mathrm{i}\,\sin\left(\frac{x}{2}\right)\,b^2\right)}{a^3\,\sin\left(\frac{x}{2}\right)\,4\mathrm{i}+b\,\cos\left(\frac{x}{2}\right)\,\left(a^2-b^2\right)\,1\mathrm{i}+a^2\,b\,\cos\left(\frac{x}{2}\right)\,1\mathrm{i}-a\,b^2\,\sin\left(\frac{x}{2}\right)\,3\mathrm{i}}\right)}{b\,\sqrt{a^2-b^2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)^2*(a + b/sin(x))),x)
```

```
[Out] log(sin(x/2)/cos(x/2))/b - (2*a*atanh(((a^2 - b^2)^(1/2)*(a^2*sin(x/2)*4i -
 b^2*sin(x/2)*1i + a*b*cos(x/2)*2i))/(a^3*sin(x/2)*4i + b*cos(x/2)*(a^2 - b
^2)*1i + a^2*b*cos(x/2)*1i - a*b^2*sin(x/2)*3i)))/(b*(a^2 - b^2)^(1/2))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{\csc^2(x)}{a+b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)**2/(a+b*csc(x)),x)
```

```
[Out] Integral(csc(x)**2/(a + b*csc(x)), x)
```

## 3.43 $\int \frac{\csc(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=40

$$-\frac{2\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}}$$

```
[Out] -2*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.07, antiderivative size = 40, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.364, Rules used = {3831, 2660, 618, 206}

$$-\frac{2\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[x]/(a + b*Csc[x]),x]
```

```
[Out] (-2*ArcTanh[(a + b*Tan[x/2])/Sqrt[a^2 - b^2]])/Sqrt[a^2 - b^2]
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx = \frac{\int \frac{1}{1+\frac{a\sin(x)}{b}}\,dx}{b}$$
$$= \frac{2\,\text{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx, x, \tan\left(\frac{x}{2}\right)\right)}{b}$$
$$= -\frac{4\,\text{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx, x, \frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b}$$
$$= -\frac{2\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}}$$

**Mathematica [A]** time = 0.02, size = 40, normalized size = 1.00

$$\frac{2\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)}{\sqrt{b^2-a^2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[x]/(a + b*Csc[x]),x]
```

```
[Out] (2*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2]
```

**fricas [A]** time = 0.53, size = 154, normalized size = 3.85

$$\left[\frac{\log\left(-\frac{\left(a^2-2b^2\right)\cos(x)^2+2ab\sin(x)+a^2+b^2-2(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)}{2\sqrt{a^2-b^2}}, -\frac{\sqrt{-a^2+b^2}\arctan\left(-\frac{\sqrt{-a^2+b^2}(b\sin(x)+a)}{\left(a^2-b^2\right)\cos(x)}\right)}{a^2-b^2}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/2*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 + b^2 - 2*(b*cos(x)*
sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b*sin(x) - a^2 - b^
2))/sqrt(a^2 - b^2), -sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) +
 a)/((a^2 - b^2)*cos(x)))/(a^2 - b^2)]
```

**giac [A]** time = 0.45, size = 48, normalized size = 1.20

$$\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\text{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{\sqrt{-a^2+b^2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] 2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
b^2)))/sqrt(-a^2 + b^2)
```

**maple [A]** time = 0.18, size = 39, normalized size = 0.98

$$\frac{2\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(x)/(a+b*csc(x)),x)
```

```
[Out] 2/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.29, size = 36, normalized size = 0.90

$$-\frac{2\,\mathrm{atanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}\,\sqrt{a-b}}\right)}{\sqrt{a+b}\,\sqrt{a-b}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(sin(x)*(a + b/sin(x))),x)
```

```
[Out] -(2*atanh((a + b*tan(x/2))/((a + b)^(1/2)*(a - b)^(1/2))))/((a + b)^(1/2)*(
a - b)^(1/2))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(x)/(a+b*csc(x)),x)
```

```
[Out] Integral(csc(x)/(a + b*csc(x)), x)
```

## 3.44 $\int \frac{1}{a+b \csc(c+dx)} \, dx$

**Optimal**. Leaf size=57

$$\frac{2b \tanh^{-1}\left(\frac{a+b \tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{ad\sqrt{a^2-b^2}}+\frac{x}{a}$$

```
[Out] x/a+2*b*arctanh((a+b*tan(1/2*d*x+1/2*c))/(a^2-b^2)^(1/2))/a/d/(a^2-b^2)^(1/
2)
```

**Rubi [A]** time = 0.06, antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.333, Rules used = {3783, 2660, 618, 206}

$$\frac{2b \tanh^{-1}\left(\frac{a+b \tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{ad\sqrt{a^2-b^2}}+\frac{x}{a}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^(-1),x]
```

```
[Out] x/a + (2*b*ArcTanh[(a + b*Tan[(c + d*x)/2])/Sqrt[a^2 - b^2]])/(a*Sqrt[a^2 -
 b^2]*d)
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3783

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Dist[1/a, Int[1/(1 + (a*Sin[c + d*x])/b), x], x] /; FreeQ[{a, b, c, d}, x
] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \frac{1}{a+b\csc(c+dx)}\,dx = \frac{x}{a} - \frac{\int \frac{1}{1+\frac{a\sin(c+dx)}{b}}\,dx}{a}$$
$$= \frac{x}{a} - \frac{2\,\text{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx, x, \tan\left(\frac{1}{2}(c+dx)\right)\right)}{ad}$$
$$= \frac{x}{a} + \frac{4\,\text{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx, x, \frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{ad}$$
$$= \frac{x}{a} + \frac{2b\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{1}{2}(c+dx)\right)\right)}{\sqrt{a^2-b^2}}\right)}{a\sqrt{a^2-b^2}\,d}$$

**Mathematica [A]** time = 0.11, size = 59, normalized size = 1.04

$$\frac{-\frac{2b\tan^{-1}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{b^2-a^2}}\right)}{d\sqrt{b^2-a^2}}+\frac{c}{d}+x}{a}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^(-1),x]
```

```
[Out] (c/d + x - (2*b*ArcTan[(a + b*Tan[(c + d*x)/2])/Sqrt[-a^2 + b^2]])/(Sqrt[-a
^2 + b^2]*d))/a
```

**fricas [A]** time = 0.61, size = 238, normalized size = 4.18

$$\left[\frac{2\left(a^2-b^2\right)dx+\sqrt{a^2-b^2}\,b\log\left(\frac{\left(a^2-2b^2\right)\cos(dx+c)^2+2ab\sin(dx+c)+a^2+b^2+2\left(b\cos(dx+c)\sin(dx+c)+a\cos(dx+c)\right)\sqrt{a^2-b^2}}{a^2\cos(dx+c)^2-2ab\sin(dx+c)-a^2-b^2}\right)}{2\left(a^3-ab^2\right)d}, \frac{\left(a^2-b\right.}{}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c)),x, algorithm="fricas")
```

```
[Out] [1/2*(2*(a^2 - b^2)*d*x + sqrt(a^2 - b^2)*b*log(((a^2 - 2*b^2)*cos(d*x + c)
^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*(b*cos(d*x + c)*sin(d*x + c) + a*co
s(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*x + c)^2 - 2*a*b*sin(d*x + c) - a^2
 - b^2)))/((a^3 - a*b^2)*d), ((a^2 - b^2)*d*x + sqrt(-a^2 + b^2)*b*arctan(-
sqrt(-a^2 + b^2)*(b*sin(d*x + c) + a)/((a^2 - b^2)*cos(d*x + c))))/((a^3 -
a*b^2)*d)]
```

**giac [A]** time = 0.32, size = 77, normalized size = 1.35

$$-\frac{\frac{2\left(\pi\left\lfloor\frac{dx+c}{2\pi}+\frac{1}{2}\right\rfloor\text{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b}{\sqrt{-a^2+b^2}\,a}-\frac{dx+c}{a}}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] -(2*(pi*floor(1/2*(d*x + c)/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*d*x + 1/2*
c) + a)/sqrt(-a^2 + b^2)))*b/(sqrt(-a^2 + b^2)*a) - (d*x + c)/a)/d
```

**maple [A]** time = 0.57, size = 70, normalized size = 1.23

$$-\frac{2b\arctan\left(\frac{2b\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{da\sqrt{-a^2+b^2}}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{da}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+b*csc(d*x+c)),x)
```

```
[Out] -2/d/a*b/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2
)^(1/2))+2/d/a*arctan(tan(1/2*d*x+1/2*c))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.45, size = 184, normalized size = 3.23

$$\frac{x}{a}-\frac{2\,b\,\mathrm{atanh}\left(\frac{2\,a^2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,\left(a^2-b^2\right)-2\,b^4\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)-2\,b^2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,\left(a^2-b^2\right)+a\,b^3\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)+3\,a^2\,b^2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)+a\,b\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,\left(a^2\right.}{a\,\left(2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^2+b\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a\right)\,\sqrt{a^2-b^2}}\right.}{a\,d\,\sqrt{a^2-b^2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + b/sin(c + d*x)),x)
```

```
[Out] x/a - (2*b*atanh((2*a^2*sin(c/2 + (d*x)/2)*(a^2 - b^2) - 2*b^4*sin(c/2 + (d
*x)/2) - 2*b^2*sin(c/2 + (d*x)/2)*(a^2 - b^2) + a*b^3*cos(c/2 + (d*x)/2) +
3*a^2*b^2*sin(c/2 + (d*x)/2) + a*b*cos(c/2 + (d*x)/2)*(a^2 - b^2))/(a*(2*a^
2*sin(c/2 + (d*x)/2) + a*b*cos(c/2 + (d*x)/2))*(a^2 - b^2)^(1/2))))/(a*d*(a
^2 - b^2)^(1/2))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{a + b \csc{\left(c + d x \right)}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c)),x)
```

```
[Out] Integral(1/(a + b*csc(c + d*x)), x)
```

# 3.45 $\int \frac{\sin(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=61

$$-\frac{2b^2\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^2\sqrt{a^2-b^2}}-\frac{bx}{a^2}-\frac{\cos(x)}{a}$$

```
[Out] -b*x/a^2-cos(x)/a-2*b^2*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/a^2/(a^2-
b^2)^(1/2)
```

**Rubi [A]** time = 0.11, antiderivative size = 61, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.546, Rules used = {3853, 12, 3783, 2660, 618, 206}

$$-\frac{2b^2\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^2\sqrt{a^2-b^2}}-\frac{bx}{a^2}-\frac{\cos(x)}{a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]/(a + b*Csc[x]),x]
```

```
[Out] -((b*x)/a^2) - (2*b^2*ArcTanh[(a + b*Tan[x/2])/Sqrt[a^2 - b^2]])/(a^2*Sqrt[
a^2 - b^2]) - Cos[x]/a
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3783

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Dist[1/a, Int[1/(1 + (a*Sin[c + d*x])/b), x], x] /; FreeQ[{a, b, c, d}, x
] && NeQ[a^2 - b^2, 0]
```

Rule 3853

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(a*f*n), x] - Dis
```

```
t[1/(a*d*n), Int[((d*Csc[e + f*x])^(n + 1)*Simp[b*n - a*(n + 1)*Csc[e + f*x
] - b*(n + 1)*Csc[e + f*x]^2, x])/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a,
 b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

Rubi steps

$$\begin{aligned}
\int \frac{\sin(x)}{a+b\csc(x)}\,dx &= -\frac{\cos(x)}{a}-\frac{\int \frac{b}{a+b\csc(x)}\,dx}{a}\\
&= -\frac{\cos(x)}{a}-\frac{b\int \frac{1}{a+b\csc(x)}\,dx}{a}\\
&= -\frac{bx}{a^2}-\frac{\cos(x)}{a}+\frac{b\int \frac{1}{1+\frac{a\sin(x)}{b}}\,dx}{a^2}\\
&= -\frac{bx}{a^2}-\frac{\cos(x)}{a}+\frac{(2b)\,\text{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{a^2}\\
&= -\frac{bx}{a^2}-\frac{\cos(x)}{a}-\frac{(4b)\,\text{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx,x,\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{a^2}\\
&= -\frac{bx}{a^2}-\frac{2b^2\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^2\sqrt{a^2-b^2}}-\frac{\cos(x)}{a}
\end{aligned}$$

**Mathematica [A]** time = 0.10, size = 56, normalized size = 0.92

$$-\frac{-\frac{2b^2\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)}{\sqrt{b^2-a^2}}+a\cos(x)+bx}{a^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]/(a + b*Csc[x]),x]
```

```
[Out] -((b*x - (2*b^2*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2]
 + a*Cos[x])/a^2)
```

**fricas [A]** time = 0.49, size = 235, normalized size = 3.85

$$\left[\frac{\sqrt{a^2-b^2}\,b^2\log\left(-\frac{\left(a^2-2b^2\right)\cos(x)^2+2ab\sin(x)+a^2+b^2-2\left(b\cos(x)\sin(x)+a\cos(x)\right)\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)-2\left(a^2b-b^3\right)x-2\left(a^3-ab^2\right)\cos}{2\left(a^4-a^2b^2\right)}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/2*(sqrt(a^2 - b^2)*b^2*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2
 + b^2 - 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*
a*b*sin(x) - a^2 - b^2)) - 2*(a^2*b - b^3)*x - 2*(a^3 - a*b^2)*cos(x))/(a^4
 - a^2*b^2), -(sqrt(-a^2 + b^2)*b^2*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) + a)
/((a^2 - b^2)*cos(x))) + (a^2*b - b^3)*x + (a^3 - a*b^2)*cos(x))/(a^4 - a^2
*b^2)]
```

**giac** [A] time = 0.52, size = 77, normalized size = 1.26

$$\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^2}{\sqrt{-a^2+b^2}\,a^2}-\frac{bx}{a^2}-\frac{2}{\left(\tan\left(\frac{1}{2}x\right)^2+1\right)a}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] 2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
b^2)))*b^2/(sqrt(-a^2 + b^2)*a^2) - b*x/a^2 - 2/((tan(1/2*x)^2 + 1)*a)
```

**maple** [A] time = 0.46, size = 72, normalized size = 1.18

$$\frac{2b^2\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{a^2\sqrt{-a^2+b^2}}-\frac{2}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)}-\frac{2b\arctan\left(\tan\left(\frac{x}{2}\right)\right)}{a^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)/(a+b*csc(x)),x)
```

```
[Out] 2*b^2/a^2/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2)
)-2/a/(tan(1/2*x)^2+1)-2/a^2*b*arctan(tan(1/2*x))
```

**maxima** [F(-2)] time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad** [B] time = 0.65, size = 766, normalized size = 12.56

$$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)^2+1\right)}-\frac{b\,x}{a^2}-\frac{b^2\,\mathrm{atan}\left(\frac{\frac{b^2\sqrt{a^2-b^2}\left(\frac{32\,b^4}{a}-\frac{32\tan\left(\frac{x}{2}\right)\left(2\,a\,b^5-2\,a^3\,b^3\right)}{a^3}+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^2\,b^2+64\,a\,b^3\tan\left(\frac{x}{2}\right)+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^3\,b^2+\frac{32\tan\left(\cdots\right.}{\cdots}\cdots\right.}{a^4-a^2\,b^2}\cdots\right.}{a^4-a^2\,b^2}\cdots\right.}{a^4-a^2\,b^2}}{\frac{128\,b^5\tan\left(\frac{x}{2}\right)}{a^3}+\frac{b^2\sqrt{a^2-b^2}\left(\frac{32\,b^4}{a}-\frac{32\tan\left(\frac{x}{2}\right)\left(2\,a\,b^5-2\,a^3\,b^3\right)}{a^3}+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^2\,b^2+64\,a\,b^3\tan\left(\frac{x}{2}\right)+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^3\,b\cdots\right.}{a^4-a^2\,b^2}\cdots\right.}{a^4-a^2\,b^2}\cdots\right.}{a^4-a^2\,b^2}}\cdots\right.}{\cdots}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)/(a + b/sin(x)),x)
```

```
[Out] - 2/(a*(tan(x/2)^2 + 1)) - (b*x)/a^2 - (b^2*atan(((b^2*(a^2 - b^2)^(1/2)*((
32*b^4)/a - (32*tan(x/2)*(2*a*b^5 - 2*a^3*b^3))/a^3 + (b^2*(a^2 - b^2)^(1/2
)*(32*a^2*b^2 + 64*a*b^3*tan(x/2) + (b^2*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (3
2*tan(x/2)*(3*a^7*b - 2*a^5*b^3))/a^3))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2))*
1i)/(a^4 - a^2*b^2) - (b^2*(a^2 - b^2)^(1/2)*((32*tan(x/2)*(2*a*b^5 - 2*a^3
*b^3))/a^3 - (32*b^4)/a + (b^2*(a^2 - b^2)^(1/2)*(32*a^2*b^2 + 64*a*b^3*tan
(x/2) - (b^2*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (32*tan(x/2)*(3*a^7*b - 2*a^5*
b^3))/a^3))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2))*1i)/(a^4 - a^2*b^2))/((128*b
^5*tan(x/2))/a^3 + (b^2*(a^2 - b^2)^(1/2)*((32*b^4)/a - (32*tan(x/2)*(2*a*b
^5 - 2*a^3*b^3))/a^3 + (b^2*(a^2 - b^2)^(1/2)*(32*a^2*b^2 + 64*a*b^3*tan(x/
2) + (b^2*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (32*tan(x/2)*(3*a^7*b - 2*a^5*b^3
))/a^3))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2) + (b^2*(a^2 -
b^2)^(1/2)*((32*tan(x/2)*(2*a*b^5 - 2*a^3*b^3))/a^3 - (32*b^4)/a + (b^2*(a^
2 - b^2)^(1/2)*(32*a^2*b^2 + 64*a*b^3*tan(x/2) - (b^2*(a^2 - b^2)^(1/2)*(32
*a^3*b^2 + (32*tan(x/2)*(3*a^7*b - 2*a^5*b^3))/a^3))/(a^4 - a^2*b^2)))/(a^4
 - a^2*b^2)))/(a^4 - a^2*b^2)))*(a^2 - b^2)^(1/2)*2i)/(a^4 - a^2*b^2)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sin(x)}{a + b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)/(a+b*csc(x)),x)
```

```
[Out] Integral(sin(x)/(a + b*csc(x)), x)
```

## 3.46 $\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=82

$$\frac{b\cos(x)}{a^2}+\frac{x\left(a^2+2b^2\right)}{2a^3}+\frac{2b^3\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^3\sqrt{a^2-b^2}}-\frac{\sin(x)\cos(x)}{2a}$$

```
[Out] 1/2*(a^2+2*b^2)*x/a^3+b*cos(x)/a^2-1/2*cos(x)*sin(x)/a+2*b^3*arctanh((a+b*t
an(1/2*x))/(a^2-b^2)^(1/2))/a^3/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.26, antiderivative size = 82, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 7, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.538, Rules used = {3853, 4104, 3919, 3831, 2660, 618, 206}

$$\frac{x\left(a^2+2b^2\right)}{2a^3}+\frac{2b^3\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^3\sqrt{a^2-b^2}}+\frac{b\cos(x)}{a^2}-\frac{\sin(x)\cos(x)}{2a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]^2/(a + b*Csc[x]),x]
```

```
[Out] ((a^2 + 2*b^2)*x)/(2*a^3) + (2*b^3*ArcTanh[(a + b*Tan[x/2])/Sqrt[a^2 - b^2]
])/(a^3*Sqrt[a^2 - b^2]) + (b*Cos[x])/a^2 - (Cos[x]*Sin[x])/(2*a)
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3853

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(a*f*n), x] - Dis
t[1/(a*d*n), Int[((d*Csc[e + f*x])^(n + 1)*Simp[b*n - a*(n + 1)*Csc[e + f*x
] - b*(n + 1)*Csc[e + f*x]^2, x])/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a,
 b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rule 4104

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a
_))^(m_), x_Symbol] :> Simp[(A*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)*(d
*Csc[e + f*x])^n)/(a*f*n), x] + Dist[1/(a*d*n), Int[(a + b*Csc[e + f*x])^m*
(d*Csc[e + f*x])^(n + 1)*Simp[a*B*n - A*b*(m + n + 1) + a*(A + A*n + C*n)*C
sc[e + f*x] + A*b*(m + n + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, d,
 e, f, A, B, C, m}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1]
```

Rubi steps

$$\begin{aligned}
\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx &= -\frac{\cos(x)\sin(x)}{2a}+\frac{\int \frac{\left(-2b+a\csc(x)+b\csc^2(x)\right)\sin(x)}{a+b\csc(x)}\,dx}{2a}\\
&= \frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}-\frac{\int \frac{-a^2-2b^2-ab\csc(x)}{a+b\csc(x)}\,dx}{2a^2}\\
&= \frac{\left(a^2+2b^2\right)x}{2a^3}+\frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}-\frac{b^3\int \frac{\csc(x)}{a+b\csc(x)}\,dx}{a^3}\\
&= \frac{\left(a^2+2b^2\right)x}{2a^3}+\frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}-\frac{b^2\int \frac{1}{1+\frac{a\sin(x)}{b}}\,dx}{a^3}\\
&= \frac{\left(a^2+2b^2\right)x}{2a^3}+\frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}-\frac{\left(2b^2\right)\text{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{a^3}\\
&= \frac{\left(a^2+2b^2\right)x}{2a^3}+\frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}+\frac{\left(4b^2\right)\text{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx,x,\frac{2a}{b}+2\tan\left(\frac{x}{2}\right.\right.}{a^3}\\
&= \frac{\left(a^2+2b^2\right)x}{2a^3}+\frac{2b^3\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^3\sqrt{a^2-b^2}}+\frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}
\end{aligned}$$

**Mathematica [A]** time = 0.11, size = 78, normalized size = 0.95

$$\frac{-\frac{8b^3\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)}{\sqrt{b^2-a^2}}+2a^2x-a^2\sin(2x)+4ab\cos(x)+4b^2x}{4a^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]^2/(a + b*Csc[x]),x]
```

```
[Out] (2*a^2*x + 4*b^2*x - (8*b^3*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt
[-a^2 + b^2] + 4*a*b*Cos[x] - a^2*Sin[2*x])/(4*a^3)
```

**fricas [A]** time = 0.68, size = 285, normalized size = 3.48

$$\left[\frac{\sqrt{a^2-b^2}\,b^3\log\left(\frac{\left(a^2-2\,b^2\right)\cos(x)^2+2\,ab\sin(x)+a^2+b^2+2\,(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right)-\left(a^4-a^2b^2\right)\cos(x)\sin(x)+\left(a^4+a^2\right.}{2\left(a^5-a^3b^2\right)}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^2/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/2*(sqrt(a^2 - b^2)*b^3*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2
+ b^2 + 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a
*b*sin(x) - a^2 - b^2)) - (a^4 - a^2*b^2)*cos(x)*sin(x) + (a^4 + a^2*b^2 -
2*b^4)*x + 2*(a^3*b - a*b^3)*cos(x))/(a^5 - a^3*b^2), 1/2*(2*sqrt(-a^2 + b^
2)*b^3*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) + a)/((a^2 - b^2)*cos(x))) - (a^4
 - a^2*b^2)*cos(x)*sin(x) + (a^4 + a^2*b^2 - 2*b^4)*x + 2*(a^3*b - a*b^3)*c
os(x))/(a^5 - a^3*b^2)]
```

**giac [A]** time = 0.64, size = 112, normalized size = 1.37

$$-\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^3}{\sqrt{-a^2+b^2}\,a^3}+\frac{\left(a^2+2\,b^2\right)x}{2\,a^3}+\frac{a\tan\left(\frac{1}{2}x\right)^3+2\,b\tan\left(\frac{1}{2}x\right)^2-a\tan\left(\frac{1}{2}x\right)+2\,b}{\left(\tan\left(\frac{1}{2}x\right)^2+1\right)^2a^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^2/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] -2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*b^3/(sqrt(-a^2 + b^2)*a^3) + 1/2*(a^2 + 2*b^2)*x/a^3 + (a*tan(1/2*x
)^3 + 2*b*tan(1/2*x)^2 - a*tan(1/2*x) + 2*b)/((tan(1/2*x)^2 + 1)^2*a^2)
```

**maple [A]** time = 0.49, size = 142, normalized size = 1.73

$$-\frac{2b^3\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{a^3\sqrt{-a^2+b^2}}+\frac{\tan^3\left(\frac{x}{2}\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}+\frac{2b\left(\tan^2\left(\frac{x}{2}\right)\right)}{a^2\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}-\frac{\tan\left(\frac{x}{2}\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}+\frac{2b}{a^2\left(\tan^2\left(\frac{x}{2}\right)+1\right)^2}+\frac{2\operatorname{arcta}}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^2/(a+b*csc(x)),x)
```

```
[Out] -2/a^3*b^3/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2
))+1/a/(tan(1/2*x)^2+1)^2*tan(1/2*x)^3+2/a^2/(tan(1/2*x)^2+1)^2*b*tan(1/2*x
)^2-1/a/(tan(1/2*x)^2+1)^2*tan(1/2*x)+2/a^2/(tan(1/2*x)^2+1)^2*b+2/a^3*arct
an(tan(1/2*x))*b^2+1/2*x/a
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^2/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.84, size = 1147, normalized size = 13.99

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^2/(a + b/sin(x)),x)
```

```
[Out] ((2*b)/a^2 - tan(x/2)/a + tan(x/2)^3/a + (2*b*tan(x/2)^2)/a^2)/(2*tan(x/2)^
2 + tan(x/2)^4 + 1) - (atan((40*b^3*tan(x/2))/(8*a^2*b + 40*b^3 + (48*b^5)/
a^2) + (48*b^5*tan(x/2))/(8*a^4*b + 48*b^5 + 40*a^2*b^3) + (8*a*b*tan(x/2))
/(8*a*b + (40*b^3)/a + (48*b^5)/a^3))*(a^2*1i + b^2*2i)*1i)/a^3 + (b^3*atan
(((b^3*(a^2 - b^2)^(1/2)*((8*(4*a^2*b^6 + 4*a^4*b^4 + a^6*b^2))/a^5 + (8*ta
n(x/2)*(2*a^8*b - 8*a^2*b^7 + 4*a^4*b^5 + 7*a^6*b^3))/a^6 + (b^3*(a^2 - b^2
)^(1/2)*(64*b^4*tan(x/2) + (8*(2*a^8*b + 2*a^6*b^3))/a^5 + (b^3*(a^2 - b^2)
^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a^10*b - 8*a^8*b^3))/a^6))/(a^5 - a^3*
b^2)))/(a^5 - a^3*b^2))*1i)/(a^5 - a^3*b^2) + (b^3*(a^2 - b^2)^(1/2)*((8*(4
*a^2*b^6 + 4*a^4*b^4 + a^6*b^2))/a^5 + (8*tan(x/2)*(2*a^8*b - 8*a^2*b^7 + 4
*a^4*b^5 + 7*a^6*b^3))/a^6 - (b^3*(a^2 - b^2)^(1/2)*(64*b^4*tan(x/2) + (8*(
2*a^8*b + 2*a^6*b^3))/a^5 - (b^3*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2
)*(12*a^10*b - 8*a^8*b^3))/a^6))/(a^5 - a^3*b^2)))/(a^5 - a^3*b^2))*1i)/(a^
5 - a^3*b^2))/((16*(2*b^7 + a^2*b^5))/a^5 + (16*tan(x/2)*(8*b^8 + 8*a^2*b^6
 + 2*a^4*b^4))/a^6 + (b^3*(a^2 - b^2)^(1/2)*((8*(4*a^2*b^6 + 4*a^4*b^4 + a^
6*b^2))/a^5 + (8*tan(x/2)*(2*a^8*b - 8*a^2*b^7 + 4*a^4*b^5 + 7*a^6*b^3))/a^
6 + (b^3*(a^2 - b^2)^(1/2)*(64*b^4*tan(x/2) + (8*(2*a^8*b + 2*a^6*b^3))/a^5
 + (b^3*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a^10*b - 8*a^8*b^3)
)/a^6))/(a^5 - a^3*b^2)))/(a^5 - a^3*b^2)))/(a^5 - a^3*b^2) - (b^3*(a^2 - b
^2)^(1/2)*((8*(4*a^2*b^6 + 4*a^4*b^4 + a^6*b^2))/a^5 + (8*tan(x/2)*(2*a^8*b
 - 8*a^2*b^7 + 4*a^4*b^5 + 7*a^6*b^3))/a^6 - (b^3*(a^2 - b^2)^(1/2)*(64*b^4
*tan(x/2) + (8*(2*a^8*b + 2*a^6*b^3))/a^5 - (b^3*(a^2 - b^2)^(1/2)*(32*a^3*
b^2 + (8*tan(x/2)*(12*a^10*b - 8*a^8*b^3))/a^6))/(a^5 - a^3*b^2)))/(a^5 - a
^3*b^2)))/(a^5 - a^3*b^2)))*(a^2 - b^2)^(1/2)*2i)/(a^5 - a^3*b^2)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sin^2(x)}{a + b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)**2/(a+b*csc(x)),x)
```

```
[Out] Integral(sin(x)**2/(a + b*csc(x)), x)
```

## 3.47 $\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=110

$$\frac{b\sin(x)\cos(x)}{2a^2}-\frac{bx\left(a^2+2b^2\right)}{2a^4}-\frac{2b^4\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^4\sqrt{a^2-b^2}}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}-\frac{\sin^2(x)\cos(x)}{3a}$$

```
[Out] -1/2*b*(a^2+2*b^2)*x/a^4-1/3*(2*a^2+3*b^2)*cos(x)/a^3+1/2*b*cos(x)*sin(x)/a
^2-1/3*cos(x)*sin(x)^2/a-2*b^4*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/a^
4/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.40, antiderivative size = 110, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.538, Rules used = {3853, 4104, 3919, 3831, 2660, 618, 206}

$$-\frac{bx\left(a^2+2b^2\right)}{2a^4}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}-\frac{2b^4\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^4\sqrt{a^2-b^2}}+\frac{b\sin(x)\cos(x)}{2a^2}-\frac{\sin^2(x)\cos(x)}{3a}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]^3/(a + b*Csc[x]),x]
```

```
[Out] -(b*(a^2 + 2*b^2)*x)/(2*a^4) - (2*b^4*ArcTanh[(a + b*Tan[x/2])/Sqrt[a^2 - b
^2]])/(a^4*Sqrt[a^2 - b^2]) - ((2*a^2 + 3*b^2)*Cos[x])/(3*a^3) + (b*Cos[x]*
Sin[x])/(2*a^2) - (Cos[x]*Sin[x]^2)/(3*a)
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3853

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(a*f*n), x] - Dis
t[1/(a*d*n), Int[((d*Csc[e + f*x])^(n + 1)*Simp[b*n - a*(n + 1)*Csc[e + f*x
] - b*(n + 1)*Csc[e + f*x]^2, x])/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a,
```

```
 b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rule 4104

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a
_))^(m_), x_Symbol] :> Simp[(A*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)*(d
*Csc[e + f*x])^n)/(a*f*n), x] + Dist[1/(a*d*n), Int[(a + b*Csc[e + f*x])^m*
(d*Csc[e + f*x])^(n + 1)*Simp[a*B*n - A*b*(m + n + 1) + a*(A + A*n + C*n)*C
sc[e + f*x] + A*b*(m + n + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, d,
 e, f, A, B, C, m}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1]
```

Rubi steps

$$
\begin{aligned}
\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx &= -\frac{\cos(x)\sin^2(x)}{3a}+\frac{\int \frac{\left(-3b+2a\csc(x)+2b\csc^2(x)\right)\sin^2(x)}{a+b\csc(x)}\,dx}{3a}\\
&=\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}-\frac{\int \frac{\left(-2\left(2a^2+3b^2\right)-ab\csc(x)+3b^2\csc^2(x)\right)\sin(x)}{a+b\csc(x)}\,dx}{6a^2}\\
&=-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}+\frac{\int \frac{-3b\left(a^2+2b^2\right)-3ab^2\csc(x)}{a+b\csc(x)}\,dx}{6a^3}\\
&=-\frac{b\left(a^2+2b^2\right)x}{2a^4}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}+\frac{b^4\int \frac{\csc(x)}{a+b\csc(x)}\ldots}{a^4}\\
&=-\frac{b\left(a^2+2b^2\right)x}{2a^4}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}+\frac{b^3\int \frac{1}{1+\frac{a\sin(x)}{b}}\ldots}{a^4}\\
&=-\frac{b\left(a^2+2b^2\right)x}{2a^4}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}+\frac{\left(2b^3\right)\mathrm{Subst}\Big(\ldots}{\ldots}\\
&=-\frac{b\left(a^2+2b^2\right)x}{2a^4}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}-\frac{\left(4b^3\right)\mathrm{Subst}\Big(\ldots}{\ldots}\\
&=-\frac{b\left(a^2+2b^2\right)x}{2a^4}-\frac{2b^4\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^4\sqrt{a^2-b^2}}-\frac{\left(2a^2+3b^2\right)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\cos\ldots
\end{aligned}
$$

**Mathematica [A]** time = 0.23, size = 98, normalized size = 0.89

$$
\frac{a^3\cos(3x)-6bx\left(a^2+2b^2\right)-3a\left(3a^2+4b^2\right)\cos(x)+\frac{24b^4\tan^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right)}{\sqrt{b^2-a^2}}+3a^2b\sin(2x)}{12a^4}
$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]^3/(a + b*Csc[x]),x]
```

```
[Out] (-6*b*(a^2 + 2*b^2)*x + (24*b^4*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/
Sqrt[-a^2 + b^2] - 3*a*(3*a^2 + 4*b^2)*Cos[x] + a^3*Cos[3*x] + 3*a^2*b*Sin[
2*x])/(12*a^4)
```

**fricas [A]** time = 0.62, size = 329, normalized size = 2.99

$$\left[\frac{3\sqrt{a^2-b^2}\,b^4\log\left(-\frac{\left(a^2-2b^2\right)\cos(x)^2+2ab\sin(x)+a^2+b^2-2\left(b\cos(x)\sin(x)+a\cos(x)\right)\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)+2\left(a^5-a^3b^2\right)\cos(x)^3+3\left(a^4b-\right.}{6\left(a^6-a^4b^2\right)}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^3/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/6*(3*sqrt(a^2 - b^2)*b^4*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a
^2 + b^2 - 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 -
2*a*b*sin(x) - a^2 - b^2)) + 2*(a^5 - a^3*b^2)*cos(x)^3 + 3*(a^4*b - a^2*b^
3)*cos(x)*sin(x) - 3*(a^4*b + a^2*b^3 - 2*b^5)*x - 6*(a^5 - a*b^4)*cos(x))/
(a^6 - a^4*b^2), -1/6*(6*sqrt(-a^2 + b^2)*b^4*arctan(-sqrt(-a^2 + b^2)*(b*s
in(x) + a)/((a^2 - b^2)*cos(x))) - 2*(a^5 - a^3*b^2)*cos(x)^3 - 3*(a^4*b -
a^2*b^3)*cos(x)*sin(x) + 3*(a^4*b + a^2*b^3 - 2*b^5)*x + 6*(a^5 - a*b^4)*co
s(x))/(a^6 - a^4*b^2)]
```

**giac [A]** time = 0.61, size = 149, normalized size = 1.35

$$\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^4}{\sqrt{-a^2+b^2}\,a^4}-\frac{\left(a^2b+2b^3\right)x}{2a^4}-\frac{3ab\tan\left(\frac{1}{2}x\right)^5+6b^2\tan\left(\frac{1}{2}x\right)^4+12a^2\tan\left(\frac{1}{2}x\right)}{3\left(\tan\left(\frac{1}{2}x\right.\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^3/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] 2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
b^2)))*b^4/(sqrt(-a^2 + b^2)*a^4) - 1/2*(a^2*b + 2*b^3)*x/a^4 - 1/3*(3*a*b*
tan(1/2*x)^5 + 6*b^2*tan(1/2*x)^4 + 12*a^2*tan(1/2*x)^2 + 12*b^2*tan(1/2*x)
^2 - 3*a*b*tan(1/2*x) + 4*a^2 + 6*b^2)/((tan(1/2*x)^2 + 1)^3*a^3)
```

**maple [B]** time = 0.44, size = 213, normalized size = 1.94

$$\frac{2b^4\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{a^4\sqrt{-a^2+b^2}}-\frac{\left(\tan^5\left(\frac{x}{2}\right)\right)b}{a^2\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}-\frac{2b^2\left(\tan^4\left(\frac{x}{2}\right)\right)}{a^3\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}-\frac{4\left(\tan^2\left(\frac{x}{2}\right)\right)}{a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}-\frac{4\left(\tan^2\left(\frac{x}{2}\right)\right)b^2}{a^3\left(\tan^2\left(\frac{x}{2}\right)+1\right)^3}+\frac{b\,\mathrm{ta}}{a^2\left(\tan^2\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^3/(a+b*csc(x)),x)
```

```
[Out] 2/a^4*b^4/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2)
)-1/a^2/(tan(1/2*x)^2+1)^3*tan(1/2*x)^5*b-2/a^3/(tan(1/2*x)^2+1)^3*b^2*tan(
1/2*x)^4-4/a/(tan(1/2*x)^2+1)^3*tan(1/2*x)^2-4/a^3/(tan(1/2*x)^2+1)^3*tan(1
/2*x)^2*b^2+1/a^2/(tan(1/2*x)^2+1)^3*b*tan(1/2*x)-4/3/a/(tan(1/2*x)^2+1)^3-
2/a^3/(tan(1/2*x)^2+1)^3*b^2-1/a^2*b*arctan(tan(1/2*x))-2/a^4*arctan(tan(1/
2*x))*b^3
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^3/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 0.95, size = 1218, normalized size = 11.07

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^3/(a + b/sin(x)),x)
```

```
[Out] - ((2*(2*a^2 + 3*b^2))/(3*a^3) + (b*tan(x/2)^5)/a^2 + (2*b^2*tan(x/2)^4)/a^
3 + (4*tan(x/2)^2*(a^2 + b^2))/a^3 - (b*tan(x/2))/a^2)/(3*tan(x/2)^2 + 3*ta
n(x/2)^4 + tan(x/2)^6 + 1) - (b^4*atan(((b^4*(a^2 - b^2)^(1/2)*((8*(4*a^3*b
^8 + 4*a^5*b^6 + a^7*b^4))/a^8 + (8*tan(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7*a^7
*b^5 + 2*a^9*b^3))/a^9 + (b^4*(a^2 - b^2)^(1/2)*((8*(2*a^8*b^4 + 2*a^10*b^2
))/a^8 + (64*b^5*tan(x/2))/a + (b^4*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(
x/2)*(12*a^13*b - 8*a^11*b^3))/a^9))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2))*1i)
/(a^6 - a^4*b^2) + (b^4*(a^2 - b^2)^(1/2)*((8*(4*a^3*b^8 + 4*a^5*b^6 + a^7*
b^4))/a^8 + (8*tan(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7*a^7*b^5 + 2*a^9*b^3))/a^
9 - (b^4*(a^2 - b^2)^(1/2)*((8*(2*a^8*b^4 + 2*a^10*b^2))/a^8 + (64*b^5*tan(
x/2))/a - (b^4*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a^13*b - 8*a
^11*b^3))/a^9))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2))*1i)/(a^6 - a^4*b^2))/((1
6*(2*b^10 + a^2*b^8))/a^8 + (16*tan(x/2)*(8*b^11 + 8*a^2*b^9 + 2*a^4*b^7))/
a^9 + (b^4*(a^2 - b^2)^(1/2)*((8*(4*a^3*b^8 + 4*a^5*b^6 + a^7*b^4))/a^8 + (
8*tan(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7*a^7*b^5 + 2*a^9*b^3))/a^9 + (b^4*(a^2
 - b^2)^(1/2)*((8*(2*a^8*b^4 + 2*a^10*b^2))/a^8 + (64*b^5*tan(x/2))/a + (b^
4*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a^13*b - 8*a^11*b^3))/a^9
))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2) - (b^4*(a^2 - b^2)^(
1/2)*((8*(4*a^3*b^8 + 4*a^5*b^6 + a^7*b^4))/a^8 + (8*tan(x/2)*(4*a^5*b^7 -
8*a^3*b^9 + 7*a^7*b^5 + 2*a^9*b^3))/a^9 - (b^4*(a^2 - b^2)^(1/2)*((8*(2*a^8
*b^4 + 2*a^10*b^2))/a^8 + (64*b^5*tan(x/2))/a - (b^4*(a^2 - b^2)^(1/2)*(32*
a^3*b^2 + (8*tan(x/2)*(12*a^13*b - 8*a^11*b^3))/a^9))/(a^6 - a^4*b^2)))/(a^
6 - a^4*b^2)))/(a^6 - a^4*b^2)))*(a^2 - b^2)^(1/2)*2i)/(a^6 - a^4*b^2) - (b
*atan((8*b^4*tan(x/2))/(8*b^4 + (40*b^6)/a^2 + (48*b^8)/a^4) + (40*b^6*tan(
x/2))/(40*b^6 + 8*a^2*b^4 + (48*b^8)/a^2) + (48*b^8*tan(x/2))/(48*b^8 + 40*
a^2*b^6 + 8*a^4*b^4))*(a^2 + 2*b^2))/a^4
```

**sympy [F(-1)]** time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)**3/(a+b*csc(x)),x)
```

```
[Out] Timed out
```

## 3.48 $\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$

**Optimal**. Leaf size=144

$$\frac{b\sin^2(x)\cos(x)}{3a^2}+\frac{2b^5\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^5\sqrt{a^2-b^2}}+\frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4}-\frac{\left(3a^2+4b^2\right)\sin(x)\cos(x)}{8a^3}+\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{8a^5}$$

```
[Out] 1/8*(3*a^4+4*a^2*b^2+8*b^4)*x/a^5+1/3*b*(2*a^2+3*b^2)*cos(x)/a^4-1/8*(3*a^2
+4*b^2)*cos(x)*sin(x)/a^3+1/3*b*cos(x)*sin(x)^2/a^2-1/4*cos(x)*sin(x)^3/a+2
*b^5*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/a^5/(a^2-b^2)^(1/2)
```

**Rubi [A]** time = 0.59, antiderivative size = 144, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 7, integrand size = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.538, Rules used = {3853, 4104, 3919, 3831, 2660, 618, 206}

$$\frac{x\left(4a^2b^2+3a^4+8b^4\right)}{8a^5}+\frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4}-\frac{\left(3a^2+4b^2\right)\sin(x)\cos(x)}{8a^3}+\frac{2b^5\tanh^{-1}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^5\sqrt{a^2-b^2}}+\frac{b\sin^2(x)\cos(x)}{3a^2}$$

Antiderivative was successfully verified.

```
[In] Int[Sin[x]^4/(a + b*Csc[x]),x]
```

```
[Out] ((3*a^4 + 4*a^2*b^2 + 8*b^4)*x)/(8*a^5) + (2*b^5*ArcTanh[(a + b*Tan[x/2])/S
qrt[a^2 - b^2]])/(a^5*Sqrt[a^2 - b^2]) + (b*(2*a^2 + 3*b^2)*Cos[x])/(3*a^4)
 - ((3*a^2 + 4*b^2)*Cos[x]*Sin[x])/(8*a^3) + (b*Cos[x]*Sin[x]^2)/(3*a^2) -
(Cos[x]*Sin[x]^3)/(4*a)
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3853

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(Cot[e + f*x]*(d*Csc[e + f*x])^n)/(a*f*n), x] - Dis
t[1/(a*d*n), Int[((d*Csc[e + f*x])^(n + 1)*Simp[b*n - a*(n + 1)*Csc[e + f*x
```

```
] - b*(n + 1)*Csc[e + f*x]^2, x])/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a,
 b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rule 4104

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a
_))^(m_), x_Symbol] :> Simp[(A*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)*(d
*Csc[e + f*x])^n)/(a*f*n), x] + Dist[1/(a*d*n), Int[(a + b*Csc[e + f*x])^m*
(d*Csc[e + f*x])^(n + 1)*Simp[a*B*n - A*b*(m + n + 1) + a*(A + A*n + C*n)*C
sc[e + f*x] + A*b*(m + n + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, d,
 e, f, A, B, C, m}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1]
```

Rubi steps

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx = -\frac{\cos(x)\sin^3(x)}{4a} + \frac{\int \frac{\left(-4b+3a\csc(x)+3b\csc^2(x)\right)\sin^3(x)}{a+b\csc(x)}\,dx}{4a}$$

$$= \frac{b\cos(x)\sin^2(x)}{3a^2} - \frac{\cos(x)\sin^3(x)}{4a} - \frac{\int \frac{\left(-3\left(3a^2+4b^2\right)-ab\csc(x)+8b^2\csc^2(x)\right)\sin^2(x)}{a+b\csc(x)}\,dx}{12a^2}$$

$$= -\frac{\left(3a^2+4b^2\right)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin^2(x)}{3a^2} - \frac{\cos(x)\sin^3(x)}{4a} + \frac{\int \frac{\left(-8b\left(2a^2+3b^2\right)+a\left(9a^2-4b^2\right.\right.}{}}{}$$

$$= \frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4} - \frac{\left(3a^2+4b^2\right)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin^2(x)}{3a^2} - \frac{\cos(x)\sin^3(x)}{4a} - \int$$

$$= \frac{\left(3a^4+4a^2b^2+8b^4\right)x}{8a^5} + \frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4} - \frac{\left(3a^2+4b^2\right)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin}{3a^2}$$

$$= \frac{\left(3a^4+4a^2b^2+8b^4\right)x}{8a^5} + \frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4} - \frac{\left(3a^2+4b^2\right)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin}{3a^2}$$

$$= \frac{\left(3a^4+4a^2b^2+8b^4\right)x}{8a^5} + \frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4} - \frac{\left(3a^2+4b^2\right)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin}{3a^2}$$

$$= \frac{\left(3a^4+4a^2b^2+8b^4\right)x}{8a^5} + \frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4} - \frac{\left(3a^2+4b^2\right)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin}{3a^2}$$

$$= \frac{\left(3a^4+4a^2b^2+8b^4\right)x}{8a^5} + \frac{2b^5\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{x}{2}\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^5\sqrt{a^2-b^2}} + \frac{b\left(2a^2+3b^2\right)\cos(x)}{3a^4} - \frac{\left(3a^2+4b^2\right)c}{8a}$$

**Mathematica [A]** time = 0.31, size = 129, normalized size = 0.90

$$\frac{36a^4x - 24a^4\sin(2x) + 3a^4\sin(4x) - 8a^3b\cos(3x) + 48a^2b^2x - 24a^2b^2\sin(2x) + 24ab\left(3a^2+4b^2\right)\cos(x) - \frac{19}{}}{96a^5}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sin[x]^4/(a + b*Csc[x]),x]
```

```
[Out] (36*a^4*x + 48*a^2*b^2*x + 96*b^4*x - (192*b^5*ArcTan[(a + b*Tan[x/2])/Sqrt
[-a^2 + b^2]])/Sqrt[-a^2 + b^2] + 24*a*b*(3*a^2 + 4*b^2)*Cos[x] - 8*a^3*b*C
os[3*x] - 24*a^4*Sin[2*x] - 24*a^2*b^2*Sin[2*x] + 3*a^4*Sin[4*x])/(96*a^5)
```

**fricas [A]** time = 1.13, size = 410, normalized size = 2.85

$$\left[\frac{12\sqrt{a^2-b^2}\,b^5\log\left(\frac{(a^2-2b^2)\cos(x)^2+2ab\sin(x)+a^2+b^2+2(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)-8\left(a^5b-a^3b^3\right)\cos(x)^3+3\left(3a^6-\right.}{24\left(a^7-\right.}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^4/(a+b*csc(x)),x, algorithm="fricas")
```

```
[Out] [1/24*(12*sqrt(a^2 - b^2)*b^5*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) +
a^2 + b^2 + 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 -
 2*a*b*sin(x) - a^2 - b^2)) - 8*(a^5*b - a^3*b^3)*cos(x)^3 + 3*(3*a^6 + a^4
*b^2 + 4*a^2*b^4 - 8*b^6)*x + 24*(a^5*b - a*b^5)*cos(x) + 3*(2*(a^6 - a^4*b
^2)*cos(x)^3 - (5*a^6 - a^4*b^2 - 4*a^2*b^4)*cos(x))*sin(x))/(a^7 - a^5*b^2
), 1/24*(24*sqrt(-a^2 + b^2)*b^5*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) + a)/((
a^2 - b^2)*cos(x))) - 8*(a^5*b - a^3*b^3)*cos(x)^3 + 3*(3*a^6 + a^4*b^2 + 4
*a^2*b^4 - 8*b^6)*x + 24*(a^5*b - a*b^5)*cos(x) + 3*(2*(a^6 - a^4*b^2)*cos(
x)^3 - (5*a^6 - a^4*b^2 - 4*a^2*b^4)*cos(x))*sin(x))/(a^7 - a^5*b^2)]
```

**giac [A]** time = 0.66, size = 252, normalized size = 1.75

$$-\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^5}{\sqrt{-a^2+b^2}\,a^5}+\frac{\left(3a^4+4a^2b^2+8b^4\right)x}{8a^5}+\frac{9a^3\tan\left(\frac{1}{2}x\right)^7+12ab^2\tan\left(\frac{1}{2}x\right)^7+2}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^4/(a+b*csc(x)),x, algorithm="giac")
```

```
[Out] -2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*b^5/(sqrt(-a^2 + b^2)*a^5) + 1/8*(3*a^4 + 4*a^2*b^2 + 8*b^4)*x/a^5
+ 1/12*(9*a^3*tan(1/2*x)^7 + 12*a*b^2*tan(1/2*x)^7 + 24*b^3*tan(1/2*x)^6 +
33*a^3*tan(1/2*x)^5 + 12*a*b^2*tan(1/2*x)^5 + 48*a^2*b*tan(1/2*x)^4 + 72*b^
3*tan(1/2*x)^4 - 33*a^3*tan(1/2*x)^3 - 12*a*b^2*tan(1/2*x)^3 + 64*a^2*b*tan
(1/2*x)^2 + 72*b^3*tan(1/2*x)^2 - 9*a^3*tan(1/2*x) - 12*a*b^2*tan(1/2*x) +
16*a^2*b + 24*b^3)/((tan(1/2*x)^2 + 1)^4*a^4)
```

**maple [B]** time = 0.38, size = 405, normalized size = 2.81

$$-\frac{2b^5\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)b+2a}{2\sqrt{-a^2+b^2}}\right)}{a^5\sqrt{-a^2+b^2}}+\frac{3\left(\tan^7\left(\frac{x}{2}\right)\right)}{4a\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4}+\frac{\left(\tan^7\left(\frac{x}{2}\right)\right)b^2}{a^3\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4}+\frac{2b^3\left(\tan^6\left(\frac{x}{2}\right)\right)}{a^4\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4}+\frac{\left(\tan^5\left(\frac{x}{2}\right)\right)b^2}{a^3\left(\tan^2\left(\frac{x}{2}\right)+1\right)^4}+\frac{11}{4a(t}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^4/(a+b*csc(x)),x)
```

```
[Out] -2*b^5/a^5/(-a^2+b^2)^(1/2)*arctan(1/2*(2*tan(1/2*x)*b+2*a)/(-a^2+b^2)^(1/2
))+3/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)^7+1/a^3/(tan(1/2*x)^2+1)^4*tan(1/2*x
```

```
)^7*b^2+2/a^4/(tan(1/2*x)^2+1)^4*b^3*tan(1/2*x)^6+1/a^3/(tan(1/2*x)^2+1)^4*
tan(1/2*x)^5*b^2+11/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)^5+4/a^2/(tan(1/2*x)^2
+1)^4*tan(1/2*x)^4*b+6/a^4/(tan(1/2*x)^2+1)^4*tan(1/2*x)^4*b^3-1/a^3/(tan(1
/2*x)^2+1)^4*tan(1/2*x)^3*b^2-11/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)^3+6/a^4/
(tan(1/2*x)^2+1)^4*tan(1/2*x)^2*b^3+16/3/a^2/(tan(1/2*x)^2+1)^4*tan(1/2*x)^
2*b-3/4/a/(tan(1/2*x)^2+1)^4*tan(1/2*x)-1/a^3/(tan(1/2*x)^2+1)^4*tan(1/2*x)
*b^2+4/3/a^2/(tan(1/2*x)^2+1)^4*b+2/a^4/(tan(1/2*x)^2+1)^4*b^3+2/a^5*arctan
(tan(1/2*x))*b^4+3/4/a*arctan(tan(1/2*x))+1/a^3*arctan(tan(1/2*x))*b^2
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)^4/(a+b*csc(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 1.26, size = 1639, normalized size = 11.38

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(x)^4/(a + b/sin(x)),x)
```

```
[Out] ((2*(2*a^2*b + 3*b^3))/(3*a^4) - (tan(x/2)*(3*a^2 + 4*b^2))/(4*a^3) + (tan(
x/2)^7*(3*a^2 + 4*b^2))/(4*a^3) - (tan(x/2)^3*(11*a^2 + 4*b^2))/(4*a^3) + (
tan(x/2)^5*(11*a^2 + 4*b^2))/(4*a^3) + (2*b^3*tan(x/2)^6)/a^4 + (2*tan(x/2)
^4*(2*a^2*b + 3*b^3))/a^4 + (2*tan(x/2)^2*(8*a^2*b + 9*b^3))/(3*a^4))/(4*ta
n(x/2)^2 + 6*tan(x/2)^4 + 4*tan(x/2)^6 + tan(x/2)^8 + 1) - (atan((81*b^3*ta
n(x/2))/(8*((27*a^2*b)/8 + (81*b^3)/8 + (63*b^5)/(2*a^2) + (35*b^7)/a^4 + (
40*b^9)/a^6)) + (63*b^5*tan(x/2))/(2*((27*a^4*b)/8 + (63*b^5)/2 + (81*a^2*b
^3)/8 + (35*b^7)/a^2 + (40*b^9)/a^4)) + (35*b^7*tan(x/2))/((27*a^6*b)/8 + 3
5*b^7 + (63*a^2*b^5)/2 + (81*a^4*b^3)/8 + (40*b^9)/a^2) + (40*b^9*tan(x/2))
/((27*a^8*b)/8 + 40*b^9 + 35*a^2*b^7 + (63*a^4*b^5)/2 + (81*a^6*b^3)/8) + (
27*a*b*tan(x/2))/(8*((27*a*b)/8 + (81*b^3)/(8*a) + (63*b^5)/(2*a^3) + (35*b
^7)/a^5 + (40*b^9)/a^7)))*(a^4*3i + b^4*8i + a^2*b^2*4i)*1i)/(4*a^5) + (b^5
*atan(((b^5*(a^2 - b^2)^(1/2)*((32*a^4*b^10 + 32*a^6*b^8 + 32*a^8*b^6 + 12*
a^10*b^4 + (9*a^12*b^2)/2)/a^11 + (tan(x/2)*(18*a^14*b - 128*a^4*b^11 + 64*
a^6*b^9 + 64*a^8*b^7 + 104*a^10*b^5 + 39*a^12*b^3))/(2*a^12) + (b^5*(a^2 -
b^2)^(1/2)*((12*a^14*b + 16*a^10*b^5 + 4*a^12*b^3)/a^11 + (64*b^6*tan(x/2))
/a^2 + (b^5*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (tan(x/2)*(192*a^16*b - 128*a^1
4*b^3))/(2*a^12)))/(a^7 - a^5*b^2)))/(a^7 - a^5*b^2))*1i)/(a^7 - a^5*b^2) +
 (b^5*(a^2 - b^2)^(1/2)*((32*a^4*b^10 + 32*a^6*b^8 + 32*a^8*b^6 + 12*a^10*b
^4 + (9*a^12*b^2)/2)/a^11 + (tan(x/2)*(18*a^14*b - 128*a^4*b^11 + 64*a^6*b^
9 + 64*a^8*b^7 + 104*a^10*b^5 + 39*a^12*b^3))/(2*a^12) - (b^5*(a^2 - b^2)^(
1/2)*((12*a^14*b + 16*a^10*b^5 + 4*a^12*b^3)/a^11 + (64*b^6*tan(x/2))/a^2 -
 (b^5*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (tan(x/2)*(192*a^16*b - 128*a^14*b^3)
)/(2*a^12)))/(a^7 - a^5*b^2)))/(a^7 - a^5*b^2))*1i)/(a^7 - a^5*b^2))/((32*b
^13 + 40*a^2*b^11 + 24*a^4*b^9 + 9*a^6*b^7)/a^11 + (tan(x/2)*(128*b^14 + 12
8*a^2*b^12 + 128*a^4*b^10 + 48*a^6*b^8 + 18*a^8*b^6))/a^12 + (b^5*(a^2 - b^
2)^(1/2)*((32*a^4*b^10 + 32*a^6*b^8 + 32*a^8*b^6 + 12*a^10*b^4 + (9*a^12*b^
2)/2)/a^11 + (tan(x/2)*(18*a^14*b - 128*a^4*b^11 + 64*a^6*b^9 + 64*a^8*b^7
+ 104*a^10*b^5 + 39*a^12*b^3))/(2*a^12) + (b^5*(a^2 - b^2)^(1/2)*((12*a^14*
b + 16*a^10*b^5 + 4*a^12*b^3)/a^11 + (64*b^6*tan(x/2))/a^2 + (b^5*(a^2 - b^
2)^(1/2)*(32*a^3*b^2 + (tan(x/2)*(192*a^16*b - 128*a^14*b^3))/(2*a^12)))/(a
^7 - a^5*b^2)))/(a^7 - a^5*b^2)))/(a^7 - a^5*b^2) - (b^5*(a^2 - b^2)^(1/2)*
```

```
((32*a^4*b^10 + 32*a^6*b^8 + 32*a^8*b^6 + 12*a^10*b^4 + (9*a^12*b^2)/2)/a^1
1 + (tan(x/2)*(18*a^14*b - 128*a^4*b^11 + 64*a^6*b^9 + 64*a^8*b^7 + 104*a^1
0*b^5 + 39*a^12*b^3))/(2*a^12) - (b^5*(a^2 - b^2)^(1/2)*((12*a^14*b + 16*a^
10*b^5 + 4*a^12*b^3)/a^11 + (64*b^6*tan(x/2))/a^2 - (b^5*(a^2 - b^2)^(1/2)*
(32*a^3*b^2 + (tan(x/2)*(192*a^16*b - 128*a^14*b^3))/(2*a^12)))/(a^7 - a^5*
b^2)))/(a^7 - a^5*b^2)))/(a^7 - a^5*b^2)))*(a^2 - b^2)^(1/2)*2i)/(a^7 - a^5
*b^2)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sin^4(x)}{a + b\csc(x)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(sin(x)**4/(a+b*csc(x)),x)
```

```
[Out] Integral(sin(x)**4/(a + b*csc(x)), x)
```

## 3.49 $\int \frac{1}{(a+b\csc(c+dx))^2}\,dx$

**Optimal**. Leaf size=108

$$\frac{2b\left(2a^2-b^2\right)\tanh^{-1}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^2d\left(a^2-b^2\right)^{3/2}}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}+\frac{x}{a^2}$$

```
[Out] x/a^2+2*b*(2*a^2-b^2)*arctanh((a+b*tan(1/2*d*x+1/2*c))/(a^2-b^2)^(1/2))/a^2
/(a^2-b^2)^(3/2)/d-b^2*cot(d*x+c)/a/(a^2-b^2)/d/(a+b*csc(d*x+c))
```

**Rubi [A]** time = 0.17, antiderivative size = 108, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {3785, 3919, 3831, 2660, 618, 206}

$$\frac{2b\left(2a^2-b^2\right)\tanh^{-1}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^2d\left(a^2-b^2\right)^{3/2}}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}+\frac{x}{a^2}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^(-2),x]
```

```
[Out] x/a^2 + (2*b*(2*a^2 - b^2)*ArcTanh[(a + b*Tan[(c + d*x)/2])/Sqrt[a^2 - b^2]
])/(a^2*(a^2 - b^2)^(3/2)*d) - (b^2*Cot[c + d*x])/(a*(a^2 - b^2)*d*(a + b*C
sc[c + d*x]))
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3785

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(b^2*Cot
[c + d*x]*(a + b*Csc[c + d*x])^(n + 1))/(a*d*(n + 1)*(a^2 - b^2)), x] + Dis
t[1/(a*(n + 1)*(a^2 - b^2)), Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 - b
^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x]
, x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Intege
rQ[2*n]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rubi steps

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx = -\frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))} - \frac{\int \frac{-a^2+b^2+ab\csc(c+dx)}{a+b\csc(c+dx)}\,dx}{a\left(a^2-b^2\right)}$$

$$= \frac{x}{a^2} - \frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))} - \frac{\left(b\left(2a^2-b^2\right)\right)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}\,dx}{a^2\left(a^2-b^2\right)}$$

$$= \frac{x}{a^2} - \frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))} - \frac{\left(2a^2-b^2\right)\int \frac{1}{1+\frac{a\sin(c+dx)}{b}}\,dx}{a^2\left(a^2-b^2\right)}$$

$$= \frac{x}{a^2} - \frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))} - \frac{\left(2\left(2a^2-b^2\right)\right)\operatorname{Subst}\left(\int \frac{1}{1+\frac{2ax}{b}+x^2}\,dx, x, \tan\left(\frac{1}{2}\right.\right.}{a^2\left(a^2-b^2\right)d}$$

$$= \frac{x}{a^2} - \frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))} + \frac{\left(4\left(2a^2-b^2\right)\right)\operatorname{Subst}\left(\int \frac{1}{-4\left(1-\frac{a^2}{b^2}\right)-x^2}\,dx, x, \frac{2a}{b}\right.}{a^2\left(a^2-b^2\right)d}$$

$$= \frac{x}{a^2} + \frac{2b\left(2a^2-b^2\right)\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{1}{2}(c+dx)\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^2\left(a^2-b^2\right)^{3/2}d} - \frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))}$$

**Mathematica [A]** time = 0.44, size = 139, normalized size = 1.29

$$\frac{\csc(c+dx)(a\sin(c+dx)+b)\left(-\frac{2b\left(b^2-2a^2\right)\tan^{-1}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{b^2-a^2}}\right)(a+b\csc(c+dx))}{\left(b^2-a^2\right)^{3/2}} + \frac{ab^2\cot(c+dx)}{(b-a)(a+b)} + (c+dx)(a+b\csc(c+dx\right.}{a^2d(a+b\csc(c+dx))^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^(-2),x]
```

```
[Out] (Csc[c + d*x]*((a*b^2*Cot[c + d*x])/((-a + b)*(a + b)) + (c + d*x)*(a + b*C
sc[c + d*x]) - (2*b*(-2*a^2 + b^2)*ArcTan[(a + b*Tan[(c + d*x)/2])/Sqrt[-a^
2 + b^2]]*(a + b*Csc[c + d*x]))/(-a^2 + b^2)^(3/2))*(b + a*Sin[c + d*x]))/(
a^2*d*(a + b*Csc[c + d*x])^2)
```

**fricas [B]** time = 0.61, size = 493, normalized size = 4.56

$$\left[\frac{2\left(a^{5}-2\,a^{3}b^{2}+ab^{4}\right)dx\sin\left(dx+c\right)+2\left(a^{4}b-2\,a^{2}b^{3}+b^{5}\right)dx+\left(2\,a^{2}b^{2}-b^{4}+\left(2\,a^{3}b-ab^{3}\right)\sin\left(dx+c\right)\right)\sqrt{a^{2}}}{2\left(\left(a^{7}-2\,a^{5}b^{2}+a^{3}b^{4}\right)d\sin\left(d\right.\right.}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^2,x, algorithm="fricas")
```

```
[Out] [1/2*(2*(a^5 - 2*a^3*b^2 + a*b^4)*d*x*sin(d*x + c) + 2*(a^4*b - 2*a^2*b^3 +
 b^5)*d*x + (2*a^2*b^2 - b^4 + (2*a^3*b - a*b^3)*sin(d*x + c))*sqrt(a^2 - b
^2)*log(((a^2 - 2*b^2)*cos(d*x + c)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*
(b*cos(d*x + c)*sin(d*x + c) + a*cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*
x + c)^2 - 2*a*b*sin(d*x + c) - a^2 - b^2)) - 2*(a^3*b^2 - a*b^4)*cos(d*x +
 c))/((a^7 - 2*a^5*b^2 + a^3*b^4)*d*sin(d*x + c) + (a^6*b - 2*a^4*b^3 + a^2
*b^5)*d), ((a^5 - 2*a^3*b^2 + a*b^4)*d*x*sin(d*x + c) + (a^4*b - 2*a^2*b^3
+ b^5)*d*x + (2*a^2*b^2 - b^4 + (2*a^3*b - a*b^3)*sin(d*x + c))*sqrt(-a^2 +
 b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(d*x + c) + a)/((a^2 - b^2)*cos(d*x +
c))) - (a^3*b^2 - a*b^4)*cos(d*x + c))/((a^7 - 2*a^5*b^2 + a^3*b^4)*d*sin(d
*x + c) + (a^6*b - 2*a^4*b^3 + a^2*b^5)*d)]
```

**giac [A]** time = 0.53, size = 158, normalized size = 1.46

$$-\frac{\frac{2\left(2\,a^{2}b-b^{3}\right)\left(\pi\left\lfloor\frac{dx+c}{2\,\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}\left(b\right)+\arctan\left(\frac{b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+a}{\sqrt{-a^{2}+b^{2}}}\right)\right)}{\left(a^{4}-a^{2}b^{2}\right)\sqrt{-a^{2}+b^{2}}}+\frac{2\left(ab\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+b^{2}\right)}{\left(a^{3}-ab^{2}\right)\left(b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^{2}+2\,a\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+b\right)}-\frac{dx+c}{a^{2}}}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^2,x, algorithm="giac")
```

```
[Out] -(2*(2*a^2*b - b^3)*(pi*floor(1/2*(d*x + c)/pi + 1/2)*sgn(b) + arctan((b*ta
n(1/2*d*x + 1/2*c) + a)/sqrt(-a^2 + b^2)))/((a^4 - a^2*b^2)*sqrt(-a^2 + b^2
)) + 2*(a*b*tan(1/2*d*x + 1/2*c) + b^2)/((a^3 - a*b^2)*(b*tan(1/2*d*x + 1/2
*c)^2 + 2*a*tan(1/2*d*x + 1/2*c) + b)) - (d*x + c)/a^2)/d
```

**maple [B]** time = 0.69, size = 247, normalized size = 2.29

$$-\frac{2b\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{d\left(\left(\tan^{2}\left(\frac{dx}{2}+\frac{c}{2}\right)\right)b+2a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+b\right)\left(a^{2}-b^{2}\right)}-\frac{2b^{2}}{da\left(\left(\tan^{2}\left(\frac{dx}{2}+\frac{c}{2}\right)\right)b+2a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+b\right)\left(a^{2}-b^{2}\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+b*csc(d*x+c))^2,x)
```

```
[Out] -2/d*b/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)/(a^2-b^2)*tan(1/2*
d*x+1/2*c)-2/d*b^2/a/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)/(a^2
-b^2)-4/d*b/(a^2-b^2)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2
*a)/(-a^2+b^2)^(1/2))+2/d*b^3/a^2/(a^2-b^2)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*
b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/2))+2/d/a^2*arctan(tan(1/2*d*x+1/2*
c))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^2,x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 4.18, size = 2677, normalized size = 24.79

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + b/sin(c + d*x))^2,x)
```

```
[Out] (b*atan(((b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a*b^6 - 2*a^3*b
^4 + a^5*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) - (32*tan(c/2 + (d*x)/2)*(2*a*b^
7 - 2*a^7*b - 8*a^3*b^5 + 9*a^5*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2) + (b*(2*a
^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a^8*b - a^6*b^3))/(a^6 + a^2*b^
4 - 2*a^4*b^2) + (32*tan(c/2 + (d*x)/2)*(2*a^4*b^6 - 6*a^6*b^4 + 4*a^8*b^2)
)/(a^7 + a^3*b^4 - 2*a^5*b^2) + (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2
)*((32*(a^5*b^6 - 2*a^7*b^4 + a^9*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (32*t
an(c/2 + (d*x)/2)*(3*a^11*b - 2*a^5*b^7 + 7*a^7*b^5 - 8*a^9*b^3))/(a^7 + a^
3*b^4 - 2*a^5*b^2)))/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))/(a^8 - a^2*b
^6 + 3*a^4*b^4 - 3*a^6*b^2))*1i)/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2) -
(b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*tan(c/2 + (d*x)/2)*(2*a*b
^7 - 2*a^7*b - 8*a^3*b^5 + 9*a^5*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2) - (32*(a
*b^6 - 2*a^3*b^4 + a^5*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (b*(2*a^2 - b^2)
*((a + b)^3*(a - b)^3)^(1/2)*((32*(a^8*b - a^6*b^3))/(a^6 + a^2*b^4 - 2*a^4
*b^2) + (32*tan(c/2 + (d*x)/2)*(2*a^4*b^6 - 6*a^6*b^4 + 4*a^8*b^2))/(a^7 +
a^3*b^4 - 2*a^5*b^2) - (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a
^5*b^6 - 2*a^7*b^4 + a^9*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (32*tan(c/2 +
(d*x)/2)*(3*a^11*b - 2*a^5*b^7 + 7*a^7*b^5 - 8*a^9*b^3))/(a^7 + a^3*b^4 - 2
*a^5*b^2)))/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))/(a^8 - a^2*b^6 + 3*a^
4*b^4 - 3*a^6*b^2))*1i)/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2))/((64*(b^5
- 2*a^2*b^3))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (64*tan(c/2 + (d*x)/2)*(2*b^6 -
 6*a^2*b^4 + 4*a^4*b^2))/(a^7 + a^3*b^4 - 2*a^5*b^2) + (b*(2*a^2 - b^2)*((a
 + b)^3*(a - b)^3)^(1/2)*((32*(a*b^6 - 2*a^3*b^4 + a^5*b^2))/(a^6 + a^2*b^4
 - 2*a^4*b^2) - (32*tan(c/2 + (d*x)/2)*(2*a*b^7 - 2*a^7*b - 8*a^3*b^5 + 9*a
^5*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2) + (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^
3)^(1/2)*((32*(a^8*b - a^6*b^3))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (32*tan(c/2
+ (d*x)/2)*(2*a^4*b^6 - 6*a^6*b^4 + 4*a^8*b^2))/(a^7 + a^3*b^4 - 2*a^5*b^2)
 + (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a^5*b^6 - 2*a^7*b^4 +
 a^9*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (32*tan(c/2 + (d*x)/2)*(3*a^11*b -
 2*a^5*b^7 + 7*a^7*b^5 - 8*a^9*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2)))/(a^8 - a
^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))
/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2) + (b*(2*a^2 - b^2)*((a + b)^3*(a -
 b)^3)^(1/2)*((32*tan(c/2 + (d*x)/2)*(2*a*b^7 - 2*a^7*b - 8*a^3*b^5 + 9*a^5
*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2) - (32*(a*b^6 - 2*a^3*b^4 + a^5*b^2))/(a^
6 + a^2*b^4 - 2*a^4*b^2) + (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((3
2*(a^8*b - a^6*b^3))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (32*tan(c/2 + (d*x)/2)*(
2*a^4*b^6 - 6*a^6*b^4 + 4*a^8*b^2))/(a^7 + a^3*b^4 - 2*a^5*b^2) - (b*(2*a^2
 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a^5*b^6 - 2*a^7*b^4 + a^9*b^2))/(
a^6 + a^2*b^4 - 2*a^4*b^2) + (32*tan(c/2 + (d*x)/2)*(3*a^11*b - 2*a^5*b^7 +
 7*a^7*b^5 - 8*a^9*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2)))/(a^8 - a^2*b^6 + 3*a
^4*b^4 - 3*a^6*b^2)))/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))/(a^8 - a^2*
b^6 + 3*a^4*b^4 - 3*a^6*b^2)))*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*2i
)/(d*(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)) - ((2*b*tan(c/2 + (d*x)/2))/(
a^2 - b^2) + (2*b^2)/(a*(a^2 - b^2)))/(d*(b + 2*a*tan(c/2 + (d*x)/2) + b*ta
```

```
n(c/2 + (d*x)/2)^2)) - (2*atan((64*a^5*b*tan(c/2 + (d*x)/2))/((64*a^3*b^9)/
(a^6 + a^2*b^4 - 2*a^4*b^2) - (192*a^5*b^7)/(a^6 + a^2*b^4 - 2*a^4*b^2) + (
128*a^7*b^5)/(a^6 + a^2*b^4 - 2*a^4*b^2) + (64*a^9*b^3)/(a^6 + a^2*b^4 - 2*
a^4*b^2) - (64*a^11*b)/(a^6 + a^2*b^4 - 2*a^4*b^2)) - (64*a*b^5*tan(c/2 + (
d*x)/2))/((64*a^3*b^9)/(a^6 + a^2*b^4 - 2*a^4*b^2) - (192*a^5*b^7)/(a^6 + a
^2*b^4 - 2*a^4*b^2) + (128*a^7*b^5)/(a^6 + a^2*b^4 - 2*a^4*b^2) + (64*a^9*b
^3)/(a^6 + a^2*b^4 - 2*a^4*b^2) - (64*a^11*b)/(a^6 + a^2*b^4 - 2*a^4*b^2))
+ (64*a^3*b^3*tan(c/2 + (d*x)/2))/((64*a^3*b^9)/(a^6 + a^2*b^4 - 2*a^4*b^2)
 - (192*a^5*b^7)/(a^6 + a^2*b^4 - 2*a^4*b^2) + (128*a^7*b^5)/(a^6 + a^2*b^4
 - 2*a^4*b^2) + (64*a^9*b^3)/(a^6 + a^2*b^4 - 2*a^4*b^2) - (64*a^11*b)/(a^6
 + a^2*b^4 - 2*a^4*b^2))))/(a^2*d)
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{(a + b\csc(c + dx))^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))**2,x)
```

```
[Out] Integral((a + b*csc(c + d*x))**(-2), x)
```

## 3.50 $\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$

**Optimal**. Leaf size=170

$$\frac{x}{a^3}-\frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2d\left(a^2-b^2\right)^2(a+b\csc(c+dx))}-\frac{b^2\cot(c+dx)}{2ad\left(a^2-b^2\right)(a+b\csc(c+dx))^2}+\frac{b\left(6a^4-5a^2b^2+2b^4\right)\tanh^{-1}\left(\frac{a+b\tan(}{\sqrt{a^2}}\right.}{a^3d\left(a^2-b^2\right)^{5/2}}$$

```
[Out] x/a^3+b*(6*a^4-5*a^2*b^2+2*b^4)*arctanh((a+b*tan(1/2*d*x+1/2*c))/(a^2-b^2)^
(1/2))/a^3/(a^2-b^2)^(5/2)/d-1/2*b^2*cot(d*x+c)/a/(a^2-b^2)/d/(a+b*csc(d*x+
c))^2-1/2*b^2*(5*a^2-2*b^2)*cot(d*x+c)/a^2/(a^2-b^2)^2/d/(a+b*csc(d*x+c))
```

**Rubi [A]** time = 0.32, antiderivative size = 170, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 7, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.583, Rules used = {3785, 4060, 3919, 3831, 2660, 618, 206}

$$\frac{b\left(-5a^2b^2+6a^4+2b^4\right)\tanh^{-1}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^3d\left(a^2-b^2\right)^{5/2}}-\frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2d\left(a^2-b^2\right)^2(a+b\csc(c+dx))}-\frac{b^2\cot(c+dx)}{2ad\left(a^2-b^2\right)(a+b\csc(c+}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^(-3),x]
```

```
[Out] x/a^3 + (b*(6*a^4 - 5*a^2*b^2 + 2*b^4)*ArcTanh[(a + b*Tan[(c + d*x)/2])/Sqr
t[a^2 - b^2]])/(a^3*(a^2 - b^2)^(5/2)*d) - (b^2*Cot[c + d*x])/(2*a*(a^2 - b
^2)*d*(a + b*Csc[c + d*x])^2) - (b^2*(5*a^2 - 2*b^2)*Cot[c + d*x])/(2*a^2*(
a^2 - b^2)^2*d*(a + b*Csc[c + d*x]))
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3785

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(b^2*Cot
[c + d*x]*(a + b*Csc[c + d*x])^(n + 1))/(a*d*(n + 1)*(a^2 - b^2)), x] + Dis
t[1/(a*(n + 1)*(a^2 - b^2)), Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 - b
^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x]
, x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Intege
rQ[2*n]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rule 4060

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_Symbol] :> Simp[((A*b^2 -
 a*b*B + a^2*C)*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1))/(a*f*(m + 1)*(a^
2 - b^2)), x] + Dist[1/(a*(m + 1)*(a^2 - b^2)), Int[(a + b*Csc[e + f*x])^(m
 + 1)*Simp[A*(a^2 - b^2)*(m + 1) - a*(A*b - a*B + b*C)*(m + 1)*Csc[e + f*x]
 + (A*b^2 - a*b*B + a^2*C)*(m + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a,
b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && LtQ[m, -1]
```

Rubi steps

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx = -\frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2} - \frac{\int \frac{-2\left(a^2-b^2\right)+2ab\csc(c+dx)-b^2\csc^2(c+dx)}{(a+b\csc(c+dx))^2}\,dx}{2a\left(a^2-b^2\right)}$$

$$= -\frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2} - \frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))} + \frac{\int \frac{2\left(a^2\right.}{}}{}$$

$$= \frac{x}{a^3} - \frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2} - \frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))} - \frac{\left(b\right.}{}$$

$$= \frac{x}{a^3} - \frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2} - \frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))} - \frac{\left(6\right.}{}$$

$$= \frac{x}{a^3} - \frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2} - \frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))} - \frac{\left(6\right.}{}$$

$$= \frac{x}{a^3} - \frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2} - \frac{b^2\left(5a^2-2b^2\right)\cot(c+dx)}{2a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))} + \frac{\left(2\right.}{}$$

$$= \frac{x}{a^3} + \frac{b\left(6a^4-5a^2b^2+2b^4\right)\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{1}{2}(c+dx)\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^3\left(a^2-b^2\right)^{5/2}d} - \frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c}$$

**Mathematica** [A] time = 1.08, size = 216, normalized size = 1.27

$$\frac{\csc^2(c+dx)(a\sin(c+dx)+b)\left(-\frac{3ab^2\left(2a^2-b^2\right)\cot(c+dx)(a\sin(c+dx)+b)}{(a-b)^2(a+b)^2}-\frac{2b\left(6a^4-5a^2b^2+2b^4\right)\csc(c+dx)(a\sin(c+dx)+b)^2\tan^{-1}\left(\frac{a+b\tan\cdots}{\sqrt{b\cdots}}\right)}{\left(b^2-a^2\right)^{5/2}}\cdots\right.}{2a^3d(a+b\csc(c+dx))^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^(-3),x]
```

```
[Out] (Csc[c + d*x]^2*(b + a*Sin[c + d*x])*((a*b^3*Cot[c + d*x])/((a - b)*(a + b)
) - (3*a*b^2*(2*a^2 - b^2)*Cot[c + d*x]*(b + a*Sin[c + d*x]))/((a - b)^2*(a
 + b)^2) + 2*(c + d*x)*Csc[c + d*x]*(b + a*Sin[c + d*x])^2 - (2*b*(6*a^4 -
5*a^2*b^2 + 2*b^4)*ArcTan[(a + b*Tan[(c + d*x)/2])/Sqrt[-a^2 + b^2]]*Csc[c
+ d*x]*(b + a*Sin[c + d*x])^2)/(-a^2 + b^2)^(5/2)))/(2*a^3*d*(a + b*Csc[c +
 d*x])^3)
```

**fricas** [B] time = 0.75, size = 933, normalized size = 5.49

$$\left[\frac{4\left(a^8-3a^6b^2+3a^4b^4-a^2b^6\right)dx\cos\left(dx+c\right)^2-4\left(a^8-2a^6b^2+2a^2b^6-b^8\right)dx-\left(6a^6b+a^4b^3-3a^2b^5+2b^7\cdots\right.}{\cdots}\right.$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^3,x, algorithm="fricas")
```

```
[Out] [1/4*(4*(a^8 - 3*a^6*b^2 + 3*a^4*b^4 - a^2*b^6)*d*x*cos(d*x + c)^2 - 4*(a^8
 - 2*a^6*b^2 + 2*a^2*b^6 - b^8)*d*x - (6*a^6*b + a^4*b^3 - 3*a^2*b^5 + 2*b^
7 - (6*a^6*b - 5*a^4*b^3 + 2*a^2*b^5)*cos(d*x + c)^2 + 2*(6*a^5*b^2 - 5*a^3
*b^4 + 2*a*b^6)*sin(d*x + c))*sqrt(a^2 - b^2)*log(((a^2 - 2*b^2)*cos(d*x +
c)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*(b*cos(d*x + c)*sin(d*x + c) + a*
cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*x + c)^2 - 2*a*b*sin(d*x + c) - a
^2 - b^2)) + 2*(5*a^5*b^3 - 7*a^3*b^5 + 2*a*b^7)*cos(d*x + c) - 2*(4*(a^7*b
 - 3*a^5*b^3 + 3*a^3*b^5 - a*b^7)*d*x - 3*(2*a^6*b^2 - 3*a^4*b^4 + a^2*b^6)
*cos(d*x + c))*sin(d*x + c))/((a^11 - 3*a^9*b^2 + 3*a^7*b^4 - a^5*b^6)*d*co
s(d*x + c)^2 - 2*(a^10*b - 3*a^8*b^3 + 3*a^6*b^5 - a^4*b^7)*d*sin(d*x + c)
- (a^11 - 2*a^9*b^2 + 2*a^5*b^6 - a^3*b^8)*d), 1/2*(2*(a^8 - 3*a^6*b^2 + 3*
a^4*b^4 - a^2*b^6)*d*x*cos(d*x + c)^2 - 2*(a^8 - 2*a^6*b^2 + 2*a^2*b^6 - b^
8)*d*x - (6*a^6*b + a^4*b^3 - 3*a^2*b^5 + 2*b^7 - (6*a^6*b - 5*a^4*b^3 + 2*
a^2*b^5)*cos(d*x + c)^2 + 2*(6*a^5*b^2 - 5*a^3*b^4 + 2*a*b^6)*sin(d*x + c))
*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(d*x + c) + a)/((a^2 - b^2
)*cos(d*x + c))) + (5*a^5*b^3 - 7*a^3*b^5 + 2*a*b^7)*cos(d*x + c) - (4*(a^7
*b - 3*a^5*b^3 + 3*a^3*b^5 - a*b^7)*d*x - 3*(2*a^6*b^2 - 3*a^4*b^4 + a^2*b^
6)*cos(d*x + c))*sin(d*x + c))/((a^11 - 3*a^9*b^2 + 3*a^7*b^4 - a^5*b^6)*d*
cos(d*x + c)^2 - 2*(a^10*b - 3*a^8*b^3 + 3*a^6*b^5 - a^4*b^7)*d*sin(d*x + c
) - (a^11 - 2*a^9*b^2 + 2*a^5*b^6 - a^3*b^8)*d)]
```

**giac** [A] time = 0.38, size = 297, normalized size = 1.75

$$-\frac{\frac{\left(6a^4b-5a^2b^3+2b^5\right)\left(\pi\left\lfloor\frac{dx+c}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{\left(a^7-2a^5b^2+a^3b^4\right)\sqrt{-a^2+b^2}}+\frac{4a^3b^2\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^3-ab^4\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^3+10a^4b\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^2+a^2b^3\tan\cdots}{\left(a^6-2a^4b^2+a^2b^4\right)\left(b\tan\cdots\right.}\cdots}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^3,x, algorithm="giac")
```

```
[Out] -((6*a^4*b - 5*a^2*b^3 + 2*b^5)*(pi*floor(1/2*(d*x + c)/pi + 1/2)*sgn(b) +
arctan((b*tan(1/2*d*x + 1/2*c) + a)/sqrt(-a^2 + b^2)))/((a^7 - 2*a^5*b^2 +
a^3*b^4)*sqrt(-a^2 + b^2)) + (4*a^3*b^2*tan(1/2*d*x + 1/2*c)^3 - a*b^4*tan(
1/2*d*x + 1/2*c)^3 + 10*a^4*b*tan(1/2*d*x + 1/2*c)^2 + a^2*b^3*tan(1/2*d*x
+ 1/2*c)^2 - 2*b^5*tan(1/2*d*x + 1/2*c)^2 + 16*a^3*b^2*tan(1/2*d*x + 1/2*c)
 - 7*a*b^4*tan(1/2*d*x + 1/2*c) + 5*a^2*b^3 - 2*b^5)/((a^6 - 2*a^4*b^2 + a^
2*b^4)*(b*tan(1/2*d*x + 1/2*c)^2 + 2*a*tan(1/2*d*x + 1/2*c) + b)^2) - (d*x
+ c)/a^3)/d
```

**maple [B]** time = 0.78, size = 796, normalized size = 4.68

$$-\frac{4a\,b^2\left(\tan^3\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{d\left(\left(\tan^2\left(\frac{dx}{2}+\frac{c}{2}\right)\right)b+2a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+b\right)^2\left(a^4-2a^2b^2+b^4\right)}+\frac{b^4\left(\tan^3\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{da\left(\left(\tan^2\left(\frac{dx}{2}+\frac{c}{2}\right)\right)b+2a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+b\right.}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+b*csc(d*x+c))^3,x)
```

```
[Out] -4/d*a*b^2/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^2/(a^4-2*a^2*b
^2+b^4)*tan(1/2*d*x+1/2*c)^3+1/d/a*b^4/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*
d*x+1/2*c)+b)^2/(a^4-2*a^2*b^2+b^4)*tan(1/2*d*x+1/2*c)^3-10/d*a^2*b/(tan(1/
2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^2/(a^4-2*a^2*b^2+b^4)*tan(1/2*d*
x+1/2*c)^2-1/d*b^3/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^2/(a^4
-2*a^2*b^2+b^4)*tan(1/2*d*x+1/2*c)^2+2/d/a^2*b^5/(tan(1/2*d*x+1/2*c)^2*b+2*
a*tan(1/2*d*x+1/2*c)+b)^2/(a^4-2*a^2*b^2+b^4)*tan(1/2*d*x+1/2*c)^2-16/d*a*b
^2/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^2/(a^4-2*a^2*b^2+b^4)*
tan(1/2*d*x+1/2*c)+7/d/a*b^4/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)
+b)^2/(a^4-2*a^2*b^2+b^4)*tan(1/2*d*x+1/2*c)-5/d*b^3/(tan(1/2*d*x+1/2*c)^2*
b+2*a*tan(1/2*d*x+1/2*c)+b)^2/(a^4-2*a^2*b^2+b^4)+2/d/a^2*b^5/(tan(1/2*d*x+
1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^2/(a^4-2*a^2*b^2+b^4)-6/d*a*b/(a^4-2*a
^2*b^2+b^4)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+
b^2)^(1/2))+5/d/a*b^3/(a^4-2*a^2*b^2+b^4)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*
tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/2))-2/d/a^3*b^5/(a^4-2*a^2*b^2+b^4)/(
-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/2))+2
/d/a^3*arctan(tan(1/2*d*x+1/2*c))
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^3,x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 8.80, size = 5917, normalized size = 34.81

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + b/sin(c + d*x))^3,x)
```

```
[Out] ((2*b^5 - 5*a^2*b^3)/(a^2*(a^4 + b^4 - 2*a^2*b^2)) + (tan(c/2 + (d*x)/2)*(7
*b^4 - 16*a^2*b^2))/(a*(a^4 + b^4 - 2*a^2*b^2)) + (tan(c/2 + (d*x)/2)^3*(b^
```

```
4 - 4*a^2*b^2))/(a*(a^4 + b^4 - 2*a^2*b^2)) - (tan(c/2 + (d*x)/2)^2*(5*a^2*
b - 2*b^3)*(2*a^2 + b^2))/(a^2*(a^4 + b^4 - 2*a^2*b^2)))/(d*(tan(c/2 + (d*x
)/2)^2*(4*a^2 + 2*b^2) + b^2*tan(c/2 + (d*x)/2)^4 + b^2 + 4*a*b*tan(c/2 + (
d*x)/2)^3 + 4*a*b*tan(c/2 + (d*x)/2))) + (2*atan((((8*(4*a^2*b^10 - 16*a^4*
b^8 + 24*a^6*b^6 - 16*a^8*b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 +
6*a^9*b^4 - 4*a^11*b^2) - (((8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*b^7 + 6*a^10*b
^5 - 8*a^12*b^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) - (
((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*a^16*b^2))/(a
^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)
*(12*a^18*b - 8*a^8*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^14*b^5 - 56*a^
16*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 +
(8*tan(c/2 + (d*x)/2)*(8*a^6*b^10 - 36*a^8*b^8 + 72*a^10*b^6 - 68*a^12*b^4
+ 24*a^14*b^2))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)
/a^3 + (8*tan(c/2 + (d*x)/2)*(8*a^12*b - 8*a^2*b^11 + 44*a^4*b^9 - 105*a^6*
b^7 + 124*a^8*b^5 - 72*a^10*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4
- 4*a^12*b^2))/a^3 + ((8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^6*b^6 - 16*a^8*b^4
 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (((
((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*a^16*b^2))/(a
^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)
*(12*a^18*b - 8*a^8*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^14*b^5 - 56*a^
16*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 +
(8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*b^7 + 6*a^10*b^5 - 8*a^12*b^3))/(a^13 + a^
5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^6*
b^10 - 36*a^8*b^8 + 72*a^10*b^6 - 68*a^12*b^4 + 24*a^14*b^2))/(a^14 + a^6*b
^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 + (8*tan(c/2 + (d*x)/2)*
(8*a^12*b - 8*a^2*b^11 + 44*a^4*b^9 - 105*a^6*b^7 + 124*a^8*b^5 - 72*a^10*b
^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))/a^3)/((16*(2*b
^9 - 13*a^2*b^7 + 26*a^4*b^5 - 24*a^6*b^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6
*a^9*b^4 - 4*a^11*b^2) - (((8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^6*b^6 - 16*a^
8*b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2)
- (((8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*b^7 + 6*a^10*b^5 - 8*a^12*b^3))/(a^13
+ a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) - (((8*(4*a^8*b^10 - 16*a^1
0*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*a^16*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^
6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^18*b - 8*a^8*b^11
 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^14*b^5 - 56*a^16*b^3))/(a^14 + a^6*b^8
 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 + (8*tan(c/2 + (d*x)/2)*(8
*a^6*b^10 - 36*a^8*b^8 + 72*a^10*b^6 - 68*a^12*b^4 + 24*a^14*b^2))/(a^14 +
a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 + (8*tan(c/2 + (d*x
)/2)*(8*a^12*b - 8*a^2*b^11 + 44*a^4*b^9 - 105*a^6*b^7 + 124*a^8*b^5 - 72*a
^10*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 +
 (((8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^6*b^6 - 16*a^8*b^4 + 4*a^10*b^2))/(a^
13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (((((8*(4*a^8*b^10 - 1
6*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*a^16*b^2))/(a^13 + a^5*b^8 - 4*a
^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^18*b - 8*a^8
*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^14*b^5 - 56*a^16*b^3))/(a^14 + a^
6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 + (8*(4*a^14*b + 2*a^
6*b^9 - 4*a^8*b^7 + 6*a^10*b^5 - 8*a^12*b^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 +
 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^6*b^10 - 36*a^8*b^8 +
 72*a^10*b^6 - 68*a^12*b^4 + 24*a^14*b^2))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*
a^10*b^4 - 4*a^12*b^2))*1i)/a^3 + (8*tan(c/2 + (d*x)/2)*(8*a^12*b - 8*a^2*b
^11 + 44*a^4*b^9 - 105*a^6*b^7 + 124*a^8*b^5 - 72*a^10*b^3))/(a^14 + a^6*b^
8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*1i)/a^3 + (16*tan(c/2 + (d*x)/2)*
(8*b^10 - 36*a^2*b^8 + 72*a^4*b^6 - 68*a^6*b^4 + 24*a^8*b^2))/(a^14 + a^6*b
^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))))/(a^3*d) + (b*atan(((b*((a + b)
^5*(a - b)^5)^(1/2)*(6*a^4 + 2*b^4 - 5*a^2*b^2)*((8*(4*a^2*b^10 - 16*a^4*b^
8 + 24*a^6*b^6 - 16*a^8*b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*
a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^12*b - 8*a^2*b^11 + 44*a
^4*b^9 - 105*a^6*b^7 + 124*a^8*b^5 - 72*a^10*b^3))/(a^14 + a^6*b^8 - 4*a^8*
b^6 + 6*a^10*b^4 - 4*a^12*b^2) - (b*((a + b)^5*(a - b)^5)^(1/2)*((8*(4*a^14
```

```
*b + 2*a^6*b^9 - 4*a^8*b^7 + 6*a^10*b^5 - 8*a^12*b^3))/(a^13 + a^5*b^8 - 4*
a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^6*b^10 - 36*
a^8*b^8 + 72*a^10*b^6 - 68*a^12*b^4 + 24*a^14*b^2))/(a^14 + a^6*b^8 - 4*a^8
*b^6 + 6*a^10*b^4 - 4*a^12*b^2) - (b*((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^1
2*b^6 - 16*a^14*b^4 + 4*a^16*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4
- 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^18*b - 8*a^8*b^11 + 44*a^10*b^9
 - 96*a^12*b^7 + 104*a^14*b^5 - 56*a^16*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 +
 6*a^10*b^4 - 4*a^12*b^2))*((a + b)^5*(a - b)^5)^(1/2)*(6*a^4 + 2*b^4 - 5*a
^2*b^2))/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*a^11
*b^2)))*(6*a^4 + 2*b^4 - 5*a^2*b^2))/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a
^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2)))*1i)/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 1
0*a^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2)) + (b*((a + b)^5*(a - b)^5)^(1/2)*(6*a
^4 + 2*b^4 - 5*a^2*b^2)*((8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^6*b^6 - 16*a^8*
b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) +
(8*tan(c/2 + (d*x)/2)*(8*a^12*b - 8*a^2*b^11 + 44*a^4*b^9 - 105*a^6*b^7 + 1
24*a^8*b^5 - 72*a^10*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^1
2*b^2) + (b*((a + b)^5*(a - b)^5)^(1/2)*((8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*b
^7 + 6*a^10*b^5 - 8*a^12*b^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*
a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^6*b^10 - 36*a^8*b^8 + 72*a^10*b^6 -
68*a^12*b^4 + 24*a^14*b^2))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^
12*b^2) + (b*((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*
a^16*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c
/2 + (d*x)/2)*(12*a^18*b - 8*a^8*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^1
4*b^5 - 56*a^16*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2
))*((a + b)^5*(a - b)^5)^(1/2)*(6*a^4 + 2*b^4 - 5*a^2*b^2))/(2*(a^13 - a^3*
b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2)))*(6*a^4 + 2*b^4 -
 5*a^2*b^2))/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*
a^11*b^2)))*1i)/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 -
 5*a^11*b^2)))/((16*(2*b^9 - 13*a^2*b^7 + 26*a^4*b^5 - 24*a^6*b^3))/(a^13 +
 a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (16*tan(c/2 + (d*x)/2)*(8*
b^10 - 36*a^2*b^8 + 72*a^4*b^6 - 68*a^6*b^4 + 24*a^8*b^2))/(a^14 + a^6*b^8
- 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2) - (b*((a + b)^5*(a - b)^5)^(1/2)*(6*
a^4 + 2*b^4 - 5*a^2*b^2)*((8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^6*b^6 - 16*a^8
*b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) +
 (8*tan(c/2 + (d*x)/2)*(8*a^12*b - 8*a^2*b^11 + 44*a^4*b^9 - 105*a^6*b^7 +
124*a^8*b^5 - 72*a^10*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^
12*b^2) - (b*((a + b)^5*(a - b)^5)^(1/2)*((8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*
b^7 + 6*a^10*b^5 - 8*a^12*b^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4
*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^6*b^10 - 36*a^8*b^8 + 72*a^10*b^6 -
 68*a^12*b^4 + 24*a^14*b^2))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a
^12*b^2) - (b*((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4
*a^16*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(
c/2 + (d*x)/2)*(12*a^18*b - 8*a^8*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^
14*b^5 - 56*a^16*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^
2))*((a + b)^5*(a - b)^5)^(1/2)*(6*a^4 + 2*b^4 - 5*a^2*b^2))/(2*(a^13 - a^3
*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2)))*(6*a^4 + 2*b^4
- 5*a^2*b^2))/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5
*a^11*b^2))))/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5
*a^11*b^2)) + (b*((a + b)^5*(a - b)^5)^(1/2)*(6*a^4 + 2*b^4 - 5*a^2*b^2)*((
8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^6*b^6 - 16*a^8*b^4 + 4*a^10*b^2))/(a^13 +
 a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a
^12*b - 8*a^2*b^11 + 44*a^4*b^9 - 105*a^6*b^7 + 124*a^8*b^5 - 72*a^10*b^3))
/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2) + (b*((a + b)^5*(a
- b)^5)^(1/2)*((8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*b^7 + 6*a^10*b^5 - 8*a^12*b
^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (
d*x)/2)*(8*a^6*b^10 - 36*a^8*b^8 + 72*a^10*b^6 - 68*a^12*b^4 + 24*a^14*b^2)
)/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2) + (b*((8*(4*a^8*b^
10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*a^16*b^2))/(a^13 + a^5*b^8
 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^18*b -
```

```
 8*a^8*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^14*b^5 - 56*a^16*b^3))/(a^1
4 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2))*((a + b)^5*(a - b)^5)^(
1/2)*(6*a^4 + 2*b^4 - 5*a^2*b^2))/(2*(a^13 - a^3*b^10 + 5*a^5*b^8 - 10*a^7*
b^6 + 10*a^9*b^4 - 5*a^11*b^2)))*(6*a^4 + 2*b^4 - 5*a^2*b^2))/(2*(a^13 - a^
3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2))))/(2*(a^13 - a^
3*b^10 + 5*a^5*b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2))))*((a + b)^5*(a
 - b)^5)^(1/2)*(6*a^4 + 2*b^4 - 5*a^2*b^2)*1i)/(d*(a^13 - a^3*b^10 + 5*a^5*
b^8 - 10*a^7*b^6 + 10*a^9*b^4 - 5*a^11*b^2))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{(a + b\csc(c + dx))^3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))**3,x)
```

```
[Out] Integral((a + b*csc(c + d*x))**(-3), x)
```

## 3.51 $\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$

**Optimal**. Leaf size=239

$$\frac{x}{a^4}-\frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2d\left(a^2-b^2\right)^2(a+b\csc(c+dx))^2}-\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}+\frac{b\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\tan}{a^4d\left(a^2-b^2\right)^{7}}$$

```
[Out] x/a^4+b*(8*a^6-8*a^4*b^2+7*a^2*b^4-2*b^6)*arctanh((a+b*tan(1/2*d*x+1/2*c))/
(a^2-b^2)^(1/2))/a^4/(a^2-b^2)^(7/2)/d-1/3*b^2*cot(d*x+c)/a/(a^2-b^2)/d/(a+
b*csc(d*x+c))^3-1/6*b^2*(8*a^2-3*b^2)*cot(d*x+c)/a^2/(a^2-b^2)^2/d/(a+b*csc
(d*x+c))^2-1/6*b^2*(26*a^4-17*a^2*b^2+6*b^4)*cot(d*x+c)/a^3/(a^2-b^2)^3/d/(
a+b*csc(d*x+c))
```

**Rubi [A]** time = 0.50, antiderivative size = 239, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.583, Rules used = {3785, 4060, 3919, 3831, 2660, 618, 206}

$$\frac{b\left(-8a^4b^2+7a^2b^4+8a^6-2b^6\right)\tanh^{-1}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^4d\left(a^2-b^2\right)^{7/2}}-\frac{b^2\left(-17a^2b^2+26a^4+6b^4\right)\cot(c+dx)}{6a^3d\left(a^2-b^2\right)^3(a+b\csc(c+dx))}-\frac{b^2\left(8a^2\right.}{6a^2d\left(a^2-b\right.}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*Csc[c + d*x])^(-4),x]
```

```
[Out] x/a^4 + (b*(8*a^6 - 8*a^4*b^2 + 7*a^2*b^4 - 2*b^6)*ArcTanh[(a + b*Tan[(c +
d*x)/2])/Sqrt[a^2 - b^2]])/(a^4*(a^2 - b^2)^(7/2)*d) - (b^2*Cot[c + d*x])/(
3*a*(a^2 - b^2)*d*(a + b*Csc[c + d*x])^3) - (b^2*(8*a^2 - 3*b^2)*Cot[c + d*
x])/(6*a^2*(a^2 - b^2)^2*d*(a + b*Csc[c + d*x])^2) - (b^2*(26*a^4 - 17*a^2*
b^2 + 6*b^4)*Cot[c + d*x])/(6*a^3*(a^2 - b^2)^3*d*(a + b*Csc[c + d*x]))
```

Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 618

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3785

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(b^2*Cot
[c + d*x]*(a + b*Csc[c + d*x])^(n + 1))/(a*d*(n + 1)*(a^2 - b^2)), x] + Dis
t[1/(a*(n + 1)*(a^2 - b^2)), Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 - b
^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x]
, x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Intege
```

```
rQ[2*n]
```

Rule 3831

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Dist[1/b, Int[1/(1 + (a*Sin[e + f*x])/b), x], x] /; FreeQ[{a, b, e, f
}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3919

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[(c*x)/a, x] - Dist[(b*c - a*d)/a, Int[Csc[e + f*x
]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
 a*d, 0]
```

Rule 4060

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_Symbol] :> Simp[((A*b^2 -
 a*b*B + a^2*C)*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1))/(a*f*(m + 1)*(a^
2 - b^2)), x] + Dist[1/(a*(m + 1)*(a^2 - b^2)), Int[(a + b*Csc[e + f*x])^(m
 + 1)*Simp[A*(a^2 - b^2)*(m + 1) - a*(A*b - a*B + b*C)*(m + 1)*Csc[e + f*x]
 + (A*b^2 - a*b*B + a^2*C)*(m + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a,
b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && LtQ[m, -1]
```

Rubi steps

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx = -\frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{\int \frac{-3\left(a^2-b^2\right)+3ab\csc(c+dx)-2b^2\csc^2(c+dx)}{(a+b\csc(c+dx))^3}\,dx}{3a\left(a^2-b^2\right)}$$

$$= -\frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))^2} + \frac{\int \frac{6(a}{}}{}$$

$$= -\frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))^2} - \frac{b^2(2}{6a^3}$$

$$= \frac{x}{a^4} - \frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))^2} - \frac{b}{}$$

$$= \frac{x}{a^4} - \frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))^2} - \frac{b}{}$$

$$= \frac{x}{a^4} - \frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))^2} - \frac{b}{}$$

$$= \frac{x}{a^4} - \frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2 d(a+b\csc(c+dx))^2} - \frac{b}{}$$

$$= \frac{x}{a^4} + \frac{b\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\tanh^{-1}\left(\frac{b\left(\frac{a}{b}+\tan\left(\frac{1}{2}(c+dx)\right)\right)}{\sqrt{a^2-b^2}}\right)}{a^4\left(a^2-b^2\right)^{7/2}d} - \frac{b^2\cot(c}{3a\left(a^2-b^2\right)d(a}$$

**Mathematica** [A] time = 2.02, size = 279, normalized size = 1.17

$$\frac{\csc^3(c+dx)(a\sin(c+dx)+b)\left(\frac{ab^3\left(12a^2-7b^2\right)\cot(c+dx)(a\sin(c+dx)+b)}{(a-b)^2(a+b)^2} - \frac{ab^2\left(36a^4-32a^2b^2+11b^4\right)\cot(c+dx)(a\sin(c+dx)+b)^2}{(a-b)^3(a+b)^3} - \frac{6b}{}\right.}{6a^4d(a+}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*Csc[c + d*x])^(-4),x]
```

```
[Out] (Csc[c + d*x]^3*(b + a*Sin[c + d*x])*((2*a*b^4*Cot[c + d*x])/((-a + b)*(a +
 b)) + (a*b^3*(12*a^2 - 7*b^2)*Cot[c + d*x]*(b + a*Sin[c + d*x]))/((a - b)^
2*(a + b)^2) - (a*b^2*(36*a^4 - 32*a^2*b^2 + 11*b^4)*Cot[c + d*x]*(b + a*Si
n[c + d*x])^2)/((a - b)^3*(a + b)^3) + 6*(c + d*x)*Csc[c + d*x]*(b + a*Sin[
c + d*x])^3 - (6*b*(-8*a^6 + 8*a^4*b^2 - 7*a^2*b^4 + 2*b^6)*ArcTan[(a + b*T
an[(c + d*x)/2])/Sqrt[-a^2 + b^2]]*Csc[c + d*x]*(b + a*Sin[c + d*x])^3)/(-a
^2 + b^2)^(7/2)))/(6*a^4*d*(a + b*Csc[c + d*x])^4)
```

**fricas** [B] time = 0.75, size = 1554, normalized size = 6.50

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^4,x, algorithm="fricas")
```

```
[Out] [1/12*(36*(a^10*b - 4*a^8*b^3 + 6*a^6*b^5 - 4*a^4*b^7 + a^2*b^9)*d*x*cos(d*
x + c)^2 - 2*(36*a^9*b^2 - 68*a^7*b^4 + 43*a^5*b^6 - 11*a^3*b^8)*cos(d*x +
c)^3 - 12*(3*a^10*b - 11*a^8*b^3 + 14*a^6*b^5 - 6*a^4*b^7 - a^2*b^9 + b^11)
*d*x - 3*(24*a^8*b^2 - 16*a^6*b^4 + 13*a^4*b^6 + a^2*b^8 - 2*b^10 - 3*(8*a^
8*b^2 - 8*a^6*b^4 + 7*a^4*b^6 - 2*a^2*b^8)*cos(d*x + c)^2 + (8*a^9*b + 16*a
^7*b^3 - 17*a^5*b^5 + 19*a^3*b^7 - 6*a*b^9 - (8*a^9*b - 8*a^7*b^3 + 7*a^5*b
^5 - 2*a^3*b^7)*cos(d*x + c)^2)*sin(d*x + c))*sqrt(a^2 - b^2)*log(((a^2 - 2
*b^2)*cos(d*x + c)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*(b*cos(d*x + c)*s
in(d*x + c) + a*cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*x + c)^2 - 2*a*b*
sin(d*x + c) - a^2 - b^2)) + 12*(6*a^9*b^2 - 7*a^7*b^4 + 2*a^3*b^8 - a*b^10
)*cos(d*x + c) + 6*(2*(a^11 - 4*a^9*b^2 + 6*a^7*b^4 - 4*a^5*b^6 + a^3*b^8)*
d*x*cos(d*x + c)^2 - 2*(a^11 - a^9*b^2 - 6*a^7*b^4 + 14*a^5*b^6 - 11*a^3*b^
8 + 3*a*b^10)*d*x + 5*(4*a^8*b^3 - 7*a^6*b^5 + 4*a^4*b^7 - a^2*b^9)*cos(d*x
 + c))*sin(d*x + c))/(3*(a^14*b - 4*a^12*b^3 + 6*a^10*b^5 - 4*a^8*b^7 + a^6
*b^9)*d*cos(d*x + c)^2 - (3*a^14*b - 11*a^12*b^3 + 14*a^10*b^5 - 6*a^8*b^7
- a^6*b^9 + a^4*b^11)*d + ((a^15 - 4*a^13*b^2 + 6*a^11*b^4 - 4*a^9*b^6 + a^
7*b^8)*d*cos(d*x + c)^2 - (a^15 - a^13*b^2 - 6*a^11*b^4 + 14*a^9*b^6 - 11*a
^7*b^8 + 3*a^5*b^10)*d)*sin(d*x + c)), 1/6*(18*(a^10*b - 4*a^8*b^3 + 6*a^6*
b^5 - 4*a^4*b^7 + a^2*b^9)*d*x*cos(d*x + c)^2 - (36*a^9*b^2 - 68*a^7*b^4 +
43*a^5*b^6 - 11*a^3*b^8)*cos(d*x + c)^3 - 6*(3*a^10*b - 11*a^8*b^3 + 14*a^6
*b^5 - 6*a^4*b^7 - a^2*b^9 + b^11)*d*x - 3*(24*a^8*b^2 - 16*a^6*b^4 + 13*a^
4*b^6 + a^2*b^8 - 2*b^10 - 3*(8*a^8*b^2 - 8*a^6*b^4 + 7*a^4*b^6 - 2*a^2*b^8
)*cos(d*x + c)^2 + (8*a^9*b + 16*a^7*b^3 - 17*a^5*b^5 + 19*a^3*b^7 - 6*a*b^
9 - (8*a^9*b - 8*a^7*b^3 + 7*a^5*b^5 - 2*a^3*b^7)*cos(d*x + c)^2)*sin(d*x +
 c))*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(d*x + c) + a)/((a^2 -
 b^2)*cos(d*x + c))) + 6*(6*a^9*b^2 - 7*a^7*b^4 + 2*a^3*b^8 - a*b^10)*cos(d
*x + c) + 3*(2*(a^11 - 4*a^9*b^2 + 6*a^7*b^4 - 4*a^5*b^6 + a^3*b^8)*d*x*cos
(d*x + c)^2 - 2*(a^11 - a^9*b^2 - 6*a^7*b^4 + 14*a^5*b^6 - 11*a^3*b^8 + 3*a
*b^10)*d*x + 5*(4*a^8*b^3 - 7*a^6*b^5 + 4*a^4*b^7 - a^2*b^9)*cos(d*x + c))*
sin(d*x + c))/(3*(a^14*b - 4*a^12*b^3 + 6*a^10*b^5 - 4*a^8*b^7 + a^6*b^9)*d
*cos(d*x + c)^2 - (3*a^14*b - 11*a^12*b^3 + 14*a^10*b^5 - 6*a^8*b^7 - a^6*b
^9 + a^4*b^11)*d + ((a^15 - 4*a^13*b^2 + 6*a^11*b^4 - 4*a^9*b^6 + a^7*b^8)*
d*cos(d*x + c)^2 - (a^15 - a^13*b^2 - 6*a^11*b^4 + 14*a^9*b^6 - 11*a^7*b^8
+ 3*a^5*b^10)*d)*sin(d*x + c))]
```

**giac [B]** time = 0.38, size = 535, normalized size = 2.24

$$-\frac{\frac{3\left(8a^6b-8a^4b^3+7a^2b^5-2b^7\right)\left(\pi\left\lfloor\frac{dx+c}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{\left(a^{10}-3a^8b^2+3a^6b^4-a^4b^6\right)\sqrt{-a^2+b^2}}+\frac{18a^5b^3\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^5-6a^3b^5\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^5+3ab^7\tan\left(\frac{1}{2}dx+\frac{1}{2}\right.}{}}{}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^4,x, algorithm="giac")
```

```
[Out] -1/3*(3*(8*a^6*b - 8*a^4*b^3 + 7*a^2*b^5 - 2*b^7)*(pi*floor(1/2*(d*x + c)/p
i + 1/2)*sgn(b) + arctan((b*tan(1/2*d*x + 1/2*c) + a)/sqrt(-a^2 + b^2)))/((
a^10 - 3*a^8*b^2 + 3*a^6*b^4 - a^4*b^6)*sqrt(-a^2 + b^2)) + (18*a^5*b^3*tan
(1/2*d*x + 1/2*c)^5 - 6*a^3*b^5*tan(1/2*d*x + 1/2*c)^5 + 3*a*b^7*tan(1/2*d*
x + 1/2*c)^5 + 84*a^6*b^2*tan(1/2*d*x + 1/2*c)^4 - 12*a^4*b^4*tan(1/2*d*x +
 1/2*c)^4 - 3*a^2*b^6*tan(1/2*d*x + 1/2*c)^4 + 6*b^8*tan(1/2*d*x + 1/2*c)^4
 + 104*a^7*b*tan(1/2*d*x + 1/2*c)^3 + 88*a^5*b^3*tan(1/2*d*x + 1/2*c)^3 - 7
8*a^3*b^5*tan(1/2*d*x + 1/2*c)^3 + 36*a*b^7*tan(1/2*d*x + 1/2*c)^3 + 228*a^
6*b^2*tan(1/2*d*x + 1/2*c)^2 - 114*a^4*b^4*tan(1/2*d*x + 1/2*c)^2 + 24*a^2*
b^6*tan(1/2*d*x + 1/2*c)^2 + 12*b^8*tan(1/2*d*x + 1/2*c)^2 + 138*a^5*b^3*ta
```

```
n(1/2*d*x + 1/2*c) - 96*a^3*b^5*tan(1/2*d*x + 1/2*c) + 33*a*b^7*tan(1/2*d*x
 + 1/2*c) + 26*a^4*b^4 - 17*a^2*b^6 + 6*b^8)/((a^9 - 3*a^7*b^2 + 3*a^5*b^4
- a^3*b^6)*(b*tan(1/2*d*x + 1/2*c)^2 + 2*a*tan(1/2*d*x + 1/2*c) + b)^3) - 3
*(d*x + c)/a^4)/d
```

**maple [B]** time = 0.82, size = 1912, normalized size = 8.00

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a+b*csc(d*x+c))^4,x)
```

```
[Out] -88/3/d*a^2*b^3/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*
a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^3-46/d*a^2*b^3/(tan(1/2*d*x+1/2*c
)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x
+1/2*c)+4/d*a*b^4/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-
3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^4+2/d/a^4*arctan(tan(1/2*d*x+1/
2*c))-8/d/a*b^6/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*
a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^2-8/d*a^2*b/(a^6-3*a^4*b^2+3*a^2*
b^4-b^6)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2
)^(1/2))-6/d*a^2*b^3/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a
^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^5+38/d*a*b^4/(tan(1/2*d*x+1/
2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*
d*x+1/2*c)^2-1/d/a^2*b^7/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^
3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^5-104/3/d*a^4*b/(tan(1/2
*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*t
an(1/2*d*x+1/2*c)^3-7/d/a^2*b^5/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)/(-a^2+b^2)^(1
/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/2))-2/d/a^3*b^8/(
tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4
-b^6)*tan(1/2*d*x+1/2*c)^4-11/d/a^2*b^7/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2
*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)-28/d*a^3*
b^2/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^
2*b^4-b^6)*tan(1/2*d*x+1/2*c)^4-4/d/a^3*b^8/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan
(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^2-26/
3/d*a*b^4/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^
2+3*a^2*b^4-b^6)+17/3/d/a*b^6/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c
)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)+2/d/a^4*b^7/(a^6-3*a^4*b^2+3*a^2*b^4-b
^6)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/
2))+1/d/a*b^6/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^
4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^4-12/d/a^2*b^7/(tan(1/2*d*x+1/2*c)^
2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1
/2*c)^3-76/d*a^3*b^2/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a
^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^2-2/d/a^3*b^8/(tan(1/2*d*x+1
/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)+8/d*b^3
/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x
+1/2*c)+2*a)/(-a^2+b^2)^(1/2))+2/d*b^5/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*
d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^5+32/d*b^5
/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b
^4-b^6)*tan(1/2*d*x+1/2*c)+26/d*b^5/(tan(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x
+1/2*c)+b)^3/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^3
```

**maxima [F(-2)]** time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))^4,x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
```

```
elp (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` for
more details)Is 4*a^2-4*b^2 positive or negative?
```

**mupad [B]** time = 12.38, size = 8167, normalized size = 34.17

result too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(a + b/sin(c + d*x))^4,x)
```

```
[Out] (2*atan((((8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8 + 60*a^11
*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*
b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) - (((8*(4*a^20*b + 2*a^8*b^13
 - 14*a^10*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18*b^3))/(
a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6
*a^18*b^2) - (((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17*b^8
+ 60*a^19*b^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 +
 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)
/2)*(12*a^25*b - 8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17*b^9
- 360*a^19*b^7 + 228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^1
0 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (8*tan
(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140*a^12*b^10 - 220*a^14*b^8 +
 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^21 + a^9*b^12 - 6*a^11*b^10
 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (8*tan(
c/2 + (d*x)/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^7*b^11 + 344*a^
9*b^9 - 396*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21 + a^9*b^12 - 6*a
^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))/a^4 + (((
(((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17*b^8 + 60*a^19*b^6
 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8
- 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^25*
b - 8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17*b^9 - 360*a^19*b^
7 + 228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b
^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (8*(4*a^20*b + 2*a^
8*b^13 - 14*a^10*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18*b
^3))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b
^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140*a
^12*b^10 - 220*a^14*b^8 + 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^21
 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^1
9*b^2))*1i)/a^4 + (8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8 +
 60*a^11*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 +
15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/
2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^7*b^11 + 344*a^9*b^9 - 396*
a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 1
5*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))/a^4)/((16*(2*b^13 - 1
1*a^2*b^11 + 34*a^4*b^9 - 66*a^6*b^7 + 64*a^8*b^5 - 48*a^10*b^3))/(a^20 + a
^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^
2) - (((8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8 + 60*a^11*b^
6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8
 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) - (((8*(4*a^20*b + 2*a^8*b^13 -
14*a^10*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18*b^3))/(a^2
0 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^
18*b^2) - (((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17*b^8 + 6
0*a^19*b^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15
*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)
*(12*a^25*b - 8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17*b^9 - 3
60*a^19*b^7 + 228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 +
 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (8*tan(c/
2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140*a^12*b^10 - 220*a^14*b^8 + 22
0*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^21 + a^9*b^12 - 6*a^11*b^10 +
15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (8*tan(c/2
```

```
 + (d*x)/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^7*b^11 + 344*a^9*b
^9 - 396*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21 + a^9*b^12 - 6*a^11
*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + ((
(((((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17*b^8 + 60*a^19*b
^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^
8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^2
5*b - 8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17*b^9 - 360*a^19*
b^7 + 228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13
*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (8*(4*a^20*b + 2*
a^8*b^13 - 14*a^10*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18
*b^3))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16
*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140
*a^12*b^10 - 220*a^14*b^8 + 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^
21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a
^19*b^2))*1i)/a^4 + (8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8
 + 60*a^11*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10
+ 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x
)/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^7*b^11 + 344*a^9*b^9 - 39
6*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 +
 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a^4 + (16*tan(c
/2 + (d*x)/2)*(8*b^14 - 52*a^2*b^12 + 140*a^4*b^10 - 220*a^6*b^8 + 220*a^8*
b^6 - 128*a^10*b^4 + 32*a^12*b^2))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13
*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))))/(a^4*d) - ((6*b^8 - 17*a^
2*b^6 + 26*a^4*b^4)/(3*a^3*(a^6 - b^6 + 3*a^2*b^4 - 3*a^4*b^2)) + (tan(c/2
+ (d*x)/2)*(11*b^7 - 32*a^2*b^5 + 46*a^4*b^3))/(a^2*(a^6 - b^6 + 3*a^2*b^4
- 3*a^4*b^2)) + (tan(c/2 + (d*x)/2)^4*(2*b^8 - a^2*b^6 - 4*a^4*b^4 + 28*a^6
*b^2))/(a^3*(a^6 - b^6 + 3*a^2*b^4 - 3*a^4*b^2)) + (2*tan(c/2 + (d*x)/2)^2*
(2*b^8 + 4*a^2*b^6 - 19*a^4*b^4 + 38*a^6*b^2))/(a^3*(a^6 - b^6 + 3*a^2*b^4
- 3*a^4*b^2)) + (2*tan(c/2 + (d*x)/2)^3*(2*a^2 + 3*b^2)*(26*a^4*b + 6*b^5 -
 17*a^2*b^3))/(3*a^2*(a^6 - b^6 + 3*a^2*b^4 - 3*a^4*b^2)) + (b^3*tan(c/2 +
(d*x)/2)^5*(6*a^4 + b^4 - 2*a^2*b^2))/(a^2*(a^6 - b^6 + 3*a^2*b^4 - 3*a^4*b
^2)))/(d*(b^3*tan(c/2 + (d*x)/2)^6 + tan(c/2 + (d*x)/2)^3*(12*a*b^2 + 8*a^3
) + tan(c/2 + (d*x)/2)^2*(12*a^2*b + 3*b^3) + tan(c/2 + (d*x)/2)^4*(12*a^2*
b + 3*b^3) + b^3 + 6*a*b^2*tan(c/2 + (d*x)/2) + 6*a*b^2*tan(c/2 + (d*x)/2)^
5)) + (b*atan(((b*((a + b)^7*(a - b)^7)^(1/2)*((8*(4*a^3*b^14 - 24*a^5*b^12
 + 60*a^7*b^10 - 80*a^9*b^8 + 60*a^11*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^2
0 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^
18*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*
a^7*b^11 + 344*a^9*b^9 - 396*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21
 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^1
9*b^2) - (b*((a + b)^7*(a - b)^7)^(1/2)*((8*(4*a^20*b + 2*a^8*b^13 - 14*a^1
0*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18*b^3))/(a^20 + a^
8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2
) + (8*tan(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140*a^12*b^10 - 220*
a^14*b^8 + 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^21 + a^9*b^12 - 6
*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2) - (b*((8
*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17*b^8 + 60*a^19*b^6 - 2
4*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20
*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^25*b -
8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17*b^9 - 360*a^19*b^7 +
228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 -
 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*((a + b)^7*(a - b)^7)^(1/2)*(8*a^
6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a
^8*b^10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2)))*(8*a^6 -
2*b^6 + 7*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b
^10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2)))*(8*a^6 - 2*b^
6 + 7*a^2*b^4 - 8*a^4*b^2)*1i)/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^
10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2)) + (b*((a + b)^7
*(a - b)^7)^(1/2)*((8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8
```

+ 60*a^11*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 +
15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)
/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^7*b^11 + 344*a^9*b^9 - 396
*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 +
15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2) + (b*((a + b)^7*(a -
b)^7)^(1/2)*((8*(4*a^20*b + 2*a^8*b^13 - 14*a^10*b^11 + 30*a^12*b^9 - 30*a^
14*b^7 + 20*a^16*b^5 - 12*a^18*b^3))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^
12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(8
*a^8*b^14 - 52*a^10*b^12 + 140*a^12*b^10 - 220*a^14*b^8 + 220*a^16*b^6 - 12
8*a^18*b^4 + 32*a^20*b^2))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 2
0*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2) + (b*((8*(4*a^11*b^14 - 24*a^13*b^12
+ 60*a^15*b^10 - 80*a^17*b^8 + 60*a^19*b^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a
^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*
a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^25*b - 8*a^11*b^15 + 60*a^13*b^13 -
192*a^15*b^11 + 340*a^17*b^9 - 360*a^19*b^7 + 228*a^21*b^5 - 80*a^23*b^3))
/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 -
6*a^19*b^2))*((a + b)^7*(a - b)^7)^(1/2)*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^
4*b^2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a
^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2)))*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^
2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a^12*
b^6 + 21*a^14*b^4 - 7*a^16*b^2)))*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^2)*1
i)/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a^12*b
^6 + 21*a^14*b^4 - 7*a^16*b^2)))/((16*(2*b^13 - 11*a^2*b^11 + 34*a^4*b^9 -
66*a^6*b^7 + 64*a^8*b^5 - 48*a^10*b^3))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15
*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (16*tan(c/2 + (d*x)/2
)*(8*b^14 - 52*a^2*b^12 + 140*a^4*b^10 - 220*a^6*b^8 + 220*a^8*b^6 - 128*a^
10*b^4 + 32*a^12*b^2))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^
15*b^6 + 15*a^17*b^4 - 6*a^19*b^2) - (b*((a + b)^7*(a - b)^7)^(1/2)*((8*(4*
a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8 + 60*a^11*b^6 - 24*a^13*b
^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^
6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^17*b - 8*a^3*b^1
5 + 60*a^5*b^13 - 189*a^7*b^11 + 344*a^9*b^9 - 396*a^11*b^7 + 272*a^13*b^5
- 116*a^15*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6
+ 15*a^17*b^4 - 6*a^19*b^2) - (b*((a + b)^7*(a - b)^7)^(1/2)*((8*(4*a^20*b
+ 2*a^8*b^13 - 14*a^10*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12
*a^18*b^3))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15
*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12
+ 140*a^12*b^10 - 220*a^14*b^8 + 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2)
)/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4
- 6*a^19*b^2) - (b*((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17
*b^8 + 60*a^19*b^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b
^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 +
(d*x)/2)*(12*a^25*b - 8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17
*b^9 - 360*a^19*b^7 + 228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^1
1*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*((a + b)^7*
(a - b)^7)^(1/2)*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^
14 + 7*a^6*b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7
*a^16*b^2)))*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 +
7*a^6*b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^1
6*b^2)))*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 + 7*a
^6*b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^
2)) + (b*((a + b)^7*(a - b)^7)^(1/2)*((8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7
*b^10 - 80*a^9*b^8 + 60*a^11*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b
^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) +
(8*tan(c/2 + (d*x)/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^7*b^11
+ 344*a^9*b^9 - 396*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21 + a^9*b^
12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2) +
(b*((a + b)^7*(a - b)^7)^(1/2)*((8*(4*a^20*b + 2*a^8*b^13 - 14*a^10*b^11 +
30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18*b^3))/(a^20 + a^8*b^12 -

```
6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*ta
n(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140*a^12*b^10 - 220*a^14*b^8
+ 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^21 + a^9*b^12 - 6*a^11*b^1
0 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2) + (b*((8*(4*a^11*
b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17*b^8 + 60*a^19*b^6 - 24*a^21*b^
4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6
 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2 + (d*x)/2)*(12*a^25*b - 8*a^11*b^
15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*a^17*b^9 - 360*a^19*b^7 + 228*a^21*
b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*
b^6 + 15*a^17*b^4 - 6*a^19*b^2))*((a + b)^7*(a - b)^7)^(1/2)*(8*a^6 - 2*b^6
 + 7*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^10 +
 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2)))*(8*a^6 - 2*b^6 + 7
*a^2*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^10 + 35*
a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2)))*(8*a^6 - 2*b^6 + 7*a^2
*b^4 - 8*a^4*b^2))/(2*(a^18 - a^4*b^14 + 7*a^6*b^12 - 21*a^8*b^10 + 35*a^10
*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2))))*((a + b)^7*(a - b)^7)^(1/
2)*(8*a^6 - 2*b^6 + 7*a^2*b^4 - 8*a^4*b^2)*1i)/(d*(a^18 - a^4*b^14 + 7*a^6*
b^12 - 21*a^8*b^10 + 35*a^10*b^8 - 35*a^12*b^6 + 21*a^14*b^4 - 7*a^16*b^2))
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{(a + b\csc(c + dx))^4}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(a+b*csc(d*x+c))**4,x)
```

```
[Out] Integral((a + b*csc(c + d*x))**(-4), x)
```

# 3.52 $\int \frac{1}{3+5\csc(c+dx)}\,dx$

**Optimal**. Leaf size=31

$$-\frac{5\tan^{-1}\left(\frac{\cos(c+dx)}{\sin(c+dx)+3}\right)}{6d}-\frac{x}{12}$$

```
[Out] -1/12*x-5/6*arctan(cos(d*x+c)/(3+sin(d*x+c)))/d
```

**Rubi [A]** time = 0.03, antiderivative size = 31, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.167, Rules used = {3783, 2657}

$$-\frac{5\tan^{-1}\left(\frac{\cos(c+dx)}{\sin(c+dx)+3}\right)}{6d}-\frac{x}{12}$$

Antiderivative was successfully verified.

```
[In] Int[(3 + 5*Csc[c + d*x])^(-1),x]
```

```
[Out] -x/12 - (5*ArcTan[Cos[c + d*x]/(3 + Sin[c + d*x])])/(6*d)
```

Rule 2657

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{q = Rt[
a^2 - b^2, 2]}, Simp[x/q, x] + Simp[(2*ArcTan[(b*Cos[c + d*x])/(a + q + b*S
in[c + d*x])])/(d*q), x]] /; FreeQ[{a, b, c, d}, x] && GtQ[a^2 - b^2, 0] &&
 PosQ[a]
```

Rule 3783

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Dist[1/a, Int[1/(1 + (a*Sin[c + d*x])/b), x], x] /; FreeQ[{a, b, c, d}, x
] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{1}{3+5\csc(c+dx)}\,dx &= \frac{x}{3}-\frac{1}{3}\int \frac{1}{1+\frac{3}{5}\sin(c+dx)}\,dx\\ &= -\frac{x}{12}-\frac{5\tan^{-1}\left(\frac{\cos(c+dx)}{3+\sin(c+dx)}\right)}{6d}\end{aligned}$$

**Mathematica [B]** time = 0.05, size = 66, normalized size = 2.13

$$\frac{2(c+dx)-5\tan^{-1}\left(\frac{2\left(\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}{\cos\left(\frac{1}{2}(c+dx)\right)-\sin\left(\frac{1}{2}(c+dx)\right)}\right)}{6d}$$

Antiderivative was successfully verified.

```
[In] Integrate[(3 + 5*Csc[c + d*x])^(-1),x]
```

```
[Out] (2*(c + d*x) - 5*ArcTan[(2*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))/(Cos[(c +
 d*x)/2] - Sin[(c + d*x)/2])])/(6*d)
```

**fricas [A]** time = 0.51, size = 33, normalized size = 1.06

$$\frac{4\,dx - 5\,\arctan\left(\frac{5\,\sin(dx+c)+3}{4\,\cos(dx+c)}\right)}{12\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(3+5*csc(d*x+c)),x, algorithm="fricas")
```

```
[Out] 1/12*(4*d*x - 5*arctan(1/4*(5*sin(d*x + c) + 3)/cos(d*x + c)))/d
```

**giac [A]** time = 0.59, size = 49, normalized size = 1.58

$$-\frac{dx + c + 10\,\arctan\left(-\frac{3\,\cos(dx+c)+\sin(dx+c)+3}{\cos(dx+c)-3\,\sin(dx+c)-9}\right)}{12\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(3+5*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] -1/12*(d*x + c + 10*arctan(-(3*cos(d*x + c) + sin(d*x + c) + 3)/(cos(d*x +
c) - 3*sin(d*x + c) - 9)))/d
```

**maple [A]** time = 0.56, size = 36, normalized size = 1.16

$$-\frac{5\arctan\left(\frac{5\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{4}+\frac{3}{4}\right)}{6d}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(3+5*csc(d*x+c)),x)
```

```
[Out] -5/6/d*arctan(5/4*tan(1/2*d*x+1/2*c)+3/4)+2/3/d*arctan(tan(1/2*d*x+1/2*c))
```

**maxima [A]** time = 0.41, size = 49, normalized size = 1.58

$$-\frac{5\,\arctan\left(\frac{5\,\sin(dx+c)}{4\,(\cos(dx+c)+1)}+\frac{3}{4}\right) - 4\,\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{6\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(3+5*csc(d*x+c)),x, algorithm="maxima")
```

```
[Out] -1/6*(5*arctan(5/4*sin(d*x + c)/(cos(d*x + c) + 1) + 3/4) - 4*arctan(sin(d*
x + c)/(cos(d*x + c) + 1)))/d
```

**mupad [B]** time = 0.25, size = 39, normalized size = 1.26

$$\frac{x}{3} - \frac{5\,\mathrm{atan}\left(\frac{7\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)-15}{24\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+20}\right)}{6\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(5/sin(c + d*x) + 3),x)
```

```
[Out] x/3 - (5*atan((7*tan(c/2 + (d*x)/2) - 15)/(24*tan(c/2 + (d*x)/2) + 20)))/(6
*d)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{5\csc(c+dx)+3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(3+5*csc(d*x+c)),x)
```

```
[Out] Integral(1/(5*csc(c + d*x) + 3), x)
```

## 3.53 $\int \frac{1}{5+3\csc(c+dx)}\,dx$

**Optimal**. Leaf size=68

$$\frac{3\log\left(\sin\left(\frac{1}{2}(c+dx)\right)+3\cos\left(\frac{1}{2}(c+dx)\right)\right)}{20d}-\frac{3\log\left(3\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}{20d}+\frac{x}{5}$$

```
[Out] 1/5*x+3/20*ln(3*cos(1/2*d*x+1/2*c)+sin(1/2*d*x+1/2*c))/d-3/20*ln(cos(1/2*d*
x+1/2*c)+3*sin(1/2*d*x+1/2*c))/d
```

**Rubi [A]** time = 0.04, antiderivative size = 68, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {3783, 2660, 616, 31}

$$\frac{3\log\left(\sin\left(\frac{1}{2}(c+dx)\right)+3\cos\left(\frac{1}{2}(c+dx)\right)\right)}{20d}-\frac{3\log\left(3\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}{20d}+\frac{x}{5}$$

Antiderivative was successfully verified.

```
[In] Int[(5 + 3*Csc[c + d*x])^(-1),x]
```

```
[Out] x/5 + (3*Log[3*Cos[(c + d*x)/2] + Sin[(c + d*x)/2]])/(20*d) - (3*Log[Cos[(c
 + d*x)/2] + 3*Sin[(c + d*x)/2]])/(20*d)
```

Rule 31

```
Int[((a_) + (b_.)*(x_))^(-1), x_Symbol] :> Simp[Log[RemoveContent[a + b*x,
x]]/b, x] /; FreeQ[{a, b}, x]
```

Rule 616

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> With[{q = Rt[b^2
 - 4*a*c, 2]}, Dist[c/q, Int[1/Simp[b/2 - q/2 + c*x, x], x], x] - Dist[c/q,
 Int[1/Simp[b/2 + q/2 + c*x, x], x], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2
- 4*a*c, 0] && PosQ[b^2 - 4*a*c] && PerfectSquareQ[b^2 - 4*a*c]
```

Rule 2660

```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Dist[(2*e)/d, Subst[Int[1/(a + 2*b*e*x + a*
e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[
a^2 - b^2, 0]
```

Rule 3783

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Dist[1/a, Int[1/(1 + (a*Sin[c + d*x])/b), x], x] /; FreeQ[{a, b, c, d}, x
] && NeQ[a^2 - b^2, 0]
```

Rule steps

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{x}{5}-\frac{1}{5}\int \frac{1}{1+\frac{5}{3}\sin(c+dx)}\,dx$$
$$= \frac{x}{5}-\frac{2\,\text{Subst}\left(\int \frac{1}{1+\frac{10x}{3}+x^2}\,dx,x,\tan\left(\frac{1}{2}(c+dx)\right)\right)}{5d}$$
$$= \frac{x}{5}-\frac{3\,\text{Subst}\left(\int \frac{1}{\frac{1}{3}+x}\,dx,x,\tan\left(\frac{1}{2}(c+dx)\right)\right)}{20d}+\frac{3\,\text{Subst}\left(\int \frac{1}{3+x}\,dx,x,\tan\left(\frac{1}{2}(c+dx)\right)\right)}{20d}$$
$$= \frac{x}{5}+\frac{3\log\left(3+\tan\left(\frac{1}{2}(c+dx)\right)\right)}{20d}-\frac{3\log\left(1+3\tan\left(\frac{1}{2}(c+dx)\right)\right)}{20d}$$

**Mathematica [A]** time = 0.05, size = 67, normalized size = 0.99

$$\frac{4(c+dx)+3\log\left(\sin\left(\frac{1}{2}(c+dx)\right)+3\cos\left(\frac{1}{2}(c+dx)\right)\right)-3\log\left(3\sin\left(\frac{1}{2}(c+dx)\right)+\cos\left(\frac{1}{2}(c+dx)\right)\right)}{20d}$$

Antiderivative was successfully verified.

```
[In] Integrate[(5 + 3*Csc[c + d*x])^(-1),x]
```

```
[Out] (4*(c + d*x) + 3*Log[3*Cos[(c + d*x)/2] + Sin[(c + d*x)/2]] - 3*Log[Cos[(c
+ d*x)/2] + 3*Sin[(c + d*x)/2]])/(20*d)
```

**fricas [A]** time = 0.54, size = 52, normalized size = 0.76

$$\frac{8\,dx+3\,\log\left(4\,\cos\left(dx+c\right)+3\,\sin\left(dx+c\right)+5\right)-3\,\log\left(-4\,\cos\left(dx+c\right)+3\,\sin\left(dx+c\right)+5\right)}{40\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(5+3*csc(d*x+c)),x, algorithm="fricas")
```

```
[Out] 1/40*(8*d*x + 3*log(4*cos(d*x + c) + 3*sin(d*x + c) + 5) - 3*log(-4*cos(d*x
 + c) + 3*sin(d*x + c) + 5))/d
```

**giac [A]** time = 0.51, size = 45, normalized size = 0.66

$$\frac{4\,dx+4\,c-3\,\log\left(\left|3\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+1\right|\right)+3\,\log\left(\left|\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+3\right|\right)}{20\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(5+3*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] 1/20*(4*d*x + 4*c - 3*log(abs(3*tan(1/2*d*x + 1/2*c) + 1)) + 3*log(abs(tan(
1/2*d*x + 1/2*c) + 3)))/d
```

**maple [A]** time = 0.64, size = 53, normalized size = 0.78

$$-\frac{3\ln\left(1+3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{20d}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{5d}+\frac{3\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3\right)}{20d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(5+3*csc(d*x+c)),x)
```

```
[Out] -3/20/d*ln(1+3*tan(1/2*d*x+1/2*c))+2/5/d*arctan(tan(1/2*d*x+1/2*c))+3/20/d*
ln(tan(1/2*d*x+1/2*c)+3)
```

**maxima [A]** time = 0.41, size = 71, normalized size = 1.04

$$\frac{8 \arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right) - 3 \log\left(\frac{3 \sin(dx+c)}{\cos(dx+c)+1} + 1\right) + 3 \log\left(\frac{\sin(dx+c)}{\cos(dx+c)+1} + 3\right)}{20\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(5+3*csc(d*x+c)),x, algorithm="maxima")
```

```
[Out] 1/20*(8*arctan(sin(d*x + c)/(cos(d*x + c) + 1)) - 3*log(3*sin(d*x + c)/(cos
(d*x + c) + 1) + 1) + 3*log(sin(d*x + c)/(cos(d*x + c) + 1) + 3))/d
```

**mupad [B]** time = 0.29, size = 27, normalized size = 0.40

$$\frac{x}{5} - \frac{3\,\mathrm{atanh}\left(\frac{1}{2\left(\frac{200\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{27}+\frac{20}{9}\right)} + \frac{41}{40}\right)}{10\,d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(3/sin(c + d*x) + 5),x)
```

```
[Out] x/5 - (3*atanh(1/(2*((200*tan(c/2 + (d*x)/2))/27 + 20/9)) + 41/40))/(10*d)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{3\csc\left(c + dx\right) + 5}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(5+3*csc(d*x+c)),x)
```

```
[Out] Integral(1/(3*csc(c + d*x) + 5), x)
```

## 3.54 $\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx$

**Optimal**. Leaf size=274

$$-\frac{\sqrt{2}\left(a^2+b^2(m+1)\right)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{b^2f(m+2)\sqrt{\csc(e+fx)+1}}$$

```
[Out] -cot(f*x+e)*(a+b*csc(f*x+e))^(1+m)/b/f/(2+m)+a*(a+b)*AppellF1(1/2,-1-m,1/2,
3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x+e))*cot(f*x+e)*(a+b*csc(f*x+e))^
m*2^(1/2)/b^2/f/(2+m)/(((a+b*csc(f*x+e))/(a+b))^m)/(1+csc(f*x+e))^(1/2)-(a^
2+b^2*(1+m))*AppellF1(1/2,-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x
+e))*cot(f*x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/b^2/f/(2+m)/(((a+b*csc(f*x+e))/(
a+b))^m)/(1+csc(f*x+e))^(1/2)
```

**Rubi [A]** time = 0.34, antiderivative size = 274, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 21, $\frac{\text{number of rules}}{\text{integrand size}} = 0.238$, Rules used = {3840, 4007, 3834, 139, 138}

$$-\frac{\sqrt{2}\left(a^2+b^2(m+1)\right)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{b^2f(m+2)\sqrt{\csc(e+fx)+1}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[e + f*x]^3*(a + b*Csc[e + f*x])^m,x]
```

```
[Out] -((Cot[e + f*x]*(a + b*Csc[e + f*x])^(1 + m))/(b*f*(2 + m))) + (Sqrt[2]*a*(
a + b)*AppellF1[1/2, 1/2, -1 - m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e
+ f*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b^2*f*(2 + m)*Sqrt[
1 + Csc[e + f*x]]*((a + b*Csc[e + f*x])/(a + b))^m) - (Sqrt[2]*(a^2 + b^2*(
1 + m))*AppellF1[1/2, 1/2, -m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e + f
*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b^2*f*(2 + m)*Sqrt[1 +
 Csc[e + f*x]]*((a + b*Csc[e + f*x])/(a + b))^m)
```

Rule 138

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Simp[((a + b*x)^(m + 1)*AppellF1[m + 1, -n, -p, m + 2,
-((d*(a + b*x))/(b*c - a*d)), -((f*(a + b*x))/(b*e - a*f))])/(b*(m + 1)*(b/
(b*c - a*d))^n*(b/(b*e - a*f))^p), x] /; FreeQ[{a, b, c, d, e, f, m, n, p},
 x] &&  !IntegerQ[m] &&  !IntegerQ[n] &&  !IntegerQ[p] && GtQ[b/(b*c - a*d)
, 0] && GtQ[b/(b*e - a*f), 0] &&  !(GtQ[d/(d*a - c*b), 0] && GtQ[d/(d*e - c
*f), 0] && SimplerQ[c + d*x, a + b*x]) &&  !(GtQ[f/(f*a - e*b), 0] && GtQ[f
/(f*c - e*d), 0] && SimplerQ[e + f*x, a + b*x])
```

Rule 139

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Dist[(e + f*x)^FracPart[p]/((b/(b*e - a*f))^IntPart[p]*
((b*(e + f*x))/(b*e - a*f))^FracPart[p]), Int[(a + b*x)^m*(c + d*x)^n*((b*e
)/(b*e - a*f) + (b*f*x)/(b*e - a*f))^p, x], x] /; FreeQ[{a, b, c, d, e, f,
m, n, p}, x] &&  !IntegerQ[m] &&  !IntegerQ[n] &&  !IntegerQ[p] && GtQ[b/(b
*c - a*d), 0] &&  !GtQ[b/(b*e - a*f), 0]
```

Rule 3834

```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_
Symbol] :> Dist[Cot[e + f*x]/(f*Sqrt[1 + Csc[e + f*x]]*Sqrt[1 - Csc[e + f*x
]]), Subst[Int[(a + b*x)^m/(Sqrt[1 + x]*Sqrt[1 - x]), x], x, Csc[e + f*x]],
```

```
x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0] &&  !IntegerQ[2*m]
```

Rule 3840

```
Int[csc[(e_.) + (f_.)*(x_)]^3*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> -Simp[(Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1))/(b*f*(m + 2)
), x] + Dist[1/(b*(m + 2)), Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*(b*(m +
 1) - a*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2
, 0] &&  !LtQ[m, -1]
```

Rule 4007

```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(cs
c[(e_.) + (f_.)*(x_)]*(B_.) + (A_)), x_Symbol] :> Dist[(A*b - a*B)/b, Int[C
sc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] + Dist[B/b, Int[Csc[e + f*x]*(a
+ b*Csc[e + f*x])^(m + 1), x], x] /; FreeQ[{a, b, A, B, e, f, m}, x] && NeQ
[A*b - a*B, 0] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx = -\frac{\cot(e+fx)(a+b\csc(e+fx))^{1+m}}{bf(2+m)} + \frac{\int \csc(e+fx)(b(1+m)-a\csc\ldots}{b(2\ldots}$$

$$= -\frac{\cot(e+fx)(a+b\csc(e+fx))^{1+m}}{bf(2+m)} - \frac{a\int \csc(e+fx)(a+b\csc(e+\ldots}{b^2(2+m)}$$

$$= -\frac{\cot(e+fx)(a+b\csc(e+fx))^{1+m}}{bf(2+m)} - \frac{(a\cot(e+fx))\,\text{Subst}\left(\int \frac{(a+bx\ldots}{\sqrt{1-x}\ldots}\right.}{b^2f(2+m)\sqrt{1-\csc(e+f\ldots}}$$

$$= -\frac{\cot(e+fx)(a+b\csc(e+fx))^{1+m}}{bf(2+m)} + \frac{\left(a(-a-b)\cot(e+fx)(a+b\,\text{c}\ldots\right.}{b\ldots}$$

$$= -\frac{\cot(e+fx)(a+b\csc(e+fx))^{1+m}}{bf(2+m)} + \frac{\sqrt{2}\,a(a+b)F_1\left(\frac{1}{2};\frac{1}{2},-1-m;\frac{3}{2}\ldots\right.}{\ldots}$$

**Mathematica [F]** time = 4.51, size = 0, normalized size = 0.00

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[Csc[e + f*x]^3*(a + b*Csc[e + f*x])^m,x]
[Out] Integrate[Csc[e + f*x]^3*(a + b*Csc[e + f*x])^m, x]
```

**fricas [F]** time = 0.62, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(b\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^3,x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x, algorithm="fricas")
[Out] integral((b*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

**giac** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^{m}\csc\left(fx+e\right)^{3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

**maple** [F] time = 2.10, size = 0, normalized size = 0.00

$$\int \left(\csc^{3}\left(fx+e\right)\right)\left(a+b\csc\left(fx+e\right)\right)^{m}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x)
[Out] int(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x)
```

**maxima** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^{m}\csc\left(fx+e\right)^{3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x, algorithm="maxima")
[Out] integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

**mupad** [F] time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\left(a+\frac{b}{\sin\left(e+f\,x\right)}\right)^{m}}{\sin\left(e+f\,x\right)^{3}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(e + f*x))^m/sin(e + f*x)^3,x)
[Out] int((a + b/sin(e + f*x))^m/sin(e + f*x)^3, x)
```

**sympy** [F] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a+b\csc\left(e+fx\right)\right)^{m}\csc^{3}\left(e+fx\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)**3*(a+b*csc(f*x+e))**m,x)
[Out] Integral((a + b*csc(e + f*x))**m*csc(e + f*x)**3, x)
```

## 3.55 $\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx$

**Optimal**. Leaf size=220

$$\frac{\sqrt{2}\,a\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{bf\sqrt{\csc(e+fx)+1}}-\frac{\sqrt{2}\,(a}{}$$

```
[Out] -(a+b)*AppellF1(1/2,-1-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x+e))
*cot(f*x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/b/f/(((a+b*csc(f*x+e))/(a+b))^m)/(1+
csc(f*x+e))^(1/2)+a*AppellF1(1/2,-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*
csc(f*x+e))*cot(f*x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/b/f/(((a+b*csc(f*x+e))/(a
+b))^m)/(1+csc(f*x+e))^(1/2)
```

**Rubi [A]** time = 0.23, antiderivative size = 220, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 4, integrand size = 21, $\frac{\text{number of rules}}{\text{integrand size}} = 0.190$, Rules used = {3838, 3834, 139, 138}

$$\frac{\sqrt{2}\,a\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{bf\sqrt{\csc(e+fx)+1}}-\frac{\sqrt{2}\,(a}{}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[e + f*x]^2*(a + b*Csc[e + f*x])^m,x]
```

```
[Out] -((Sqrt[2]*(a + b)*AppellF1[1/2, 1/2, -1 - m, 3/2, (1 - Csc[e + f*x])/2, (b
*(1 - Csc[e + f*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b*f*Sqr
t[1 + Csc[e + f*x]]*((a + b*Csc[e + f*x])/(a + b))^m)) + (Sqrt[2]*a*AppellF
1[1/2, 1/2, -m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e + f*x]))/(a + b)]*
Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b*f*Sqrt[1 + Csc[e + f*x]]*((a + b*Cs
c[e + f*x])/(a + b))^m)
```

Rule 138

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Simp[((a + b*x)^(m + 1)*AppellF1[m + 1, -n, -p, m + 2,
-((d*(a + b*x))/(b*c - a*d)), -((f*(a + b*x))/(b*e - a*f))])/(b*(m + 1)*(b/
(b*c - a*d))^n*(b/(b*e - a*f))^p), x] /; FreeQ[{a, b, c, d, e, f, m, n, p},
 x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[b/(b*c - a*d)
, 0] && GtQ[b/(b*e - a*f), 0] && !(GtQ[d/(d*a - c*b), 0] && GtQ[d/(d*e - c
*f), 0] && SimplerQ[c + d*x, a + b*x]) && !(GtQ[f/(f*a - e*b), 0] && GtQ[f
/(f*c - e*d), 0] && SimplerQ[e + f*x, a + b*x])
```

Rule 139

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Dist[(e + f*x)^FracPart[p]/((b/(b*e - a*f))^IntPart[p]*
((b*(e + f*x))/(b*e - a*f))^FracPart[p]), Int[(a + b*x)^m*(c + d*x)^n*((b*e
)/(b*e - a*f) + (b*f*x)/(b*e - a*f))^p, x], x] /; FreeQ[{a, b, c, d, e, f,
m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[b/(b
*c - a*d), 0] && !GtQ[b/(b*e - a*f), 0]
```

Rule 3834

```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_
Symbol] :> Dist[Cot[e + f*x]/(f*Sqrt[1 + Csc[e + f*x]]*Sqrt[1 - Csc[e + f*x
]]), Subst[Int[(a + b*x)^m/(Sqrt[1 + x]*Sqrt[1 - x]), x], x, Csc[e + f*x]],
 x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0] && !IntegerQ[2*m]
```

Rule 3838

```
Int[csc[(e_.) + (f_.)*(x_)]^2*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> -Dist[a/b, Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] + D
ist[1/b, Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^(m + 1), x], x] /; FreeQ[{a,
 b, e, f, m}, x] && NeQ[a^2 - b^2, 0]
```

Rubi steps

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx = \frac{\int \csc(e+fx)(a+b\csc(e+fx))^{1+m}\,dx}{b} - \frac{a\int \csc(e+fx)(a+b\csc(e+
$$

$$= \frac{\cot(e+fx)\,\text{Subst}\left(\int \frac{(a+bx)^{1+m}}{\sqrt{1-x}\sqrt{1+x}}\,dx, x, \csc(e+fx)\right)}{bf\sqrt{1-\csc(e+fx)}\sqrt{1+\csc(e+fx)}} - \frac{(a\cot(e+fx))\,\text{Su}}{bf\sqrt{1-\text{cs}}}$$

$$= -\frac{\left(a\cot(e+fx)(a+b\csc(e+fx))^m\left(-\frac{a+b\csc(e+fx)}{-a-b}\right)^{-m}\right)\text{Subst}\left(\int \frac{\left(-\frac{a}{-a-b}\right.}{\sqrt{1-x}}\right.}{bf\sqrt{1-\csc(e+fx)}\sqrt{1+\csc(e+fx)}}$$

$$= -\frac{\sqrt{2}\,(a+b)F_1\left(\frac{1}{2};\frac{1}{2},-1-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+}{bf\sqrt{1+\csc(e+fx)}}$$

**Mathematica [F]** time = 2.77, size = 0, normalized size = 0.00

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[Csc[e + f*x]^2*(a + b*Csc[e + f*x])^m,x]
[Out] Integrate[Csc[e + f*x]^2*(a + b*Csc[e + f*x])^m, x]
```

**fricas [F]** time = 0.53, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(b\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^2, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x, algorithm="fricas")
[Out] integral((b*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^m\csc\left(fx+e\right)^2\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

**maple [F]** time = 1.32, size = 0, normalized size = 0.00

$$\int \left(\csc^2\left(fx+e\right)\right)\left(a+b\csc\left(fx+e\right)\right)^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x)
```

```
[Out] int(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b \csc \left(fx + e\right) + a\right)^{m} \csc \left(fx + e\right)^{2} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

```
[Out] integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\left(a + \frac{b}{\sin\left(e+f\,x\right)}\right)^{m}}{\sin\left(e + f\,x\right)^{2}} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(e + f*x))^m/sin(e + f*x)^2,x)
```

```
[Out] int((a + b/sin(e + f*x))^m/sin(e + f*x)^2, x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a + b \csc \left(e + fx\right)\right)^{m} \csc^{2}\left(e + fx\right)\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)**2*(a+b*csc(f*x+e))**m,x)
```

```
[Out] Integral((a + b*csc(e + f*x))**m*csc(e + f*x)**2, x)
```

# 3.56 $\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx$

**Optimal**. Leaf size=104

$$-\frac{\sqrt{2}\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{f\sqrt{\csc(e+fx)+1}}$$

```
[Out] -AppellF1(1/2,-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x+e))*cot(f*x
+e)*(a+b*csc(f*x+e))^m*2^(1/2)/f/(((a+b*csc(f*x+e))/(a+b))^m)/(1+csc(f*x+e)
)^(1/2)
```

**Rubi [A]** time = 0.07, antiderivative size = 104, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 19, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.158, Rules used = {3834, 139, 138}

$$-\frac{\sqrt{2}\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{f\sqrt{\csc(e+fx)+1}}$$

Antiderivative was successfully verified.

```
[In] Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m,x]
```

```
[Out] -((Sqrt[2]*AppellF1[1/2, 1/2, -m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e
+ f*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(f*Sqrt[1 + Csc[e +
f*x]]*((a + b*Csc[e + f*x])/(a + b))^m))
```

Rule 138

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Simp[((a + b*x)^(m + 1)*AppellF1[m + 1, -n, -p, m + 2,
-((d*(a + b*x))/(b*c - a*d)), -((f*(a + b*x))/(b*e - a*f))])/(b*(m + 1)*(b/
(b*c - a*d))^n*(b/(b*e - a*f))^p), x] /; FreeQ[{a, b, c, d, e, f, m, n, p},
 x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[b/(b*c - a*d)
, 0] && GtQ[b/(b*e - a*f), 0] && !(GtQ[d/(d*a - c*b), 0] && GtQ[d/(d*e - c
*f), 0] && SimplerQ[c + d*x, a + b*x]) && !(GtQ[f/(f*a - e*b), 0] && GtQ[f
/(f*c - e*d), 0] && SimplerQ[e + f*x, a + b*x])
```

Rule 139

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_Symbol] :> Dist[(e + f*x)^FracPart[p]/((b/(b*e - a*f))^IntPart[p]*
((b*(e + f*x))/(b*e - a*f))^FracPart[p]), Int[(a + b*x)^m*(c + d*x)^n*((b*e
)/(b*e - a*f) + (b*f*x)/(b*e - a*f))^p, x], x] /; FreeQ[{a, b, c, d, e, f,
m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[b/(b
*c - a*d), 0] && !GtQ[b/(b*e - a*f), 0]
```

Rule 3834

```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_
Symbol] :> Dist[Cot[e + f*x]/(f*Sqrt[1 + Csc[e + f*x]]*Sqrt[1 - Csc[e + f*x
]]), Subst[Int[(a + b*x)^m/(Sqrt[1 + x]*Sqrt[1 - x]), x], x, Csc[e + f*x]],
 x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0] && !IntegerQ[2*m]
```

Rubi steps

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \frac{\cot(e+fx)\operatorname{Subst}\left(\int \frac{(a+bx)^m}{\sqrt{1-x}\sqrt{1+x}}\,dx, x, \csc(e+fx)\right)}{f\sqrt{1-\csc(e+fx)}\sqrt{1+\csc(e+fx)}}$$

$$= \frac{\left(\cot(e+fx)(a+b\csc(e+fx))^m\left(-\frac{a+b\csc(e+fx)}{-a-b}\right)^{-m}\right)\operatorname{Subst}\left(\int \frac{\left(-\frac{a}{-a-b}-\right.}{\sqrt{1-x}\,\sqrt{}}\right.}{f\sqrt{1-\csc(e+fx)}\sqrt{1+\csc(e+fx)}}$$

$$= -\frac{\sqrt{2}F_1\left(\frac{1}{2};\frac{1}{2},-m;\frac{3}{2};\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+fx)(a+}{f\sqrt{1+\csc(e+fx)}}$$

**Mathematica [F]** time = 1.97, size = 0, normalized size = 0.00

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[Csc[e + f*x]*(a + b*Csc[e + f*x])^m,x]
[Out] Integrate[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x]
```

**fricas [F]** time = 0.51, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(b\csc\left(fx+e\right)+a\right)^{m}\csc\left(fx+e\right),x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+b*csc(f*x+e))^m,x, algorithm="fricas")
[Out] integral((b*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

**giac [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^{m}\csc\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+b*csc(f*x+e))^m,x, algorithm="giac")
[Out] integrate((b*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

**maple [F]** time = 1.72, size = 0, normalized size = 0.00

$$\int \csc\left(fx+e\right)\left(a+b\csc\left(fx+e\right)\right)^{m}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(csc(f*x+e)*(a+b*csc(f*x+e))^m,x)
[Out] int(csc(f*x+e)*(a+b*csc(f*x+e))^m,x)
```

**maxima [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^{m}\csc\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

```
[Out] integrate((b*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

**mupad [F]** time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\left(a+\frac{b}{\sin\left(e+f\,x\right)}\right)^m}{\sin\left(e+f\,x\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(e + f*x))^m/sin(e + f*x),x)
```

```
[Out] int((a + b/sin(e + f*x))^m/sin(e + f*x), x)
```

**sympy [F]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a + b \csc\left(e + f x\right)\right)^{m} \csc\left(e + f x\right)\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(f*x+e)*(a+b*csc(f*x+e))**m,x)
```

```
[Out] Integral((a + b*csc(e + f*x))**m*csc(e + f*x), x)
```

# 3.57 $\int (a + b\csc(e + fx))^m \, dx$

**Optimal**. Leaf size=15

$$\text{Int}\left((a + b\csc(e + fx))^m, x\right)$$

```
[Out] Unintegrable((a+b*csc(f*x+e))^m,x)
```

**Rubi [A]** time = 0.01, antiderivative size = 0, normalized size of antiderivative = 0.00, number of steps used = 0, number of rules used = 0, integrand size = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {}

$$\int (a + b\csc(e + fx))^m \, dx$$

Verification is Not applicable to the result.

```
[In] Int[(a + b*Csc[e + f*x])^m,x]
```

```
[Out] Defer[Int][(a + b*Csc[e + f*x])^m, x]
```

Rubi steps

$$\int (a + b\csc(e + fx))^m \, dx = \int (a + b\csc(e + fx))^m \, dx$$

**Mathematica [A]** time = 1.50, size = 0, normalized size = 0.00

$$\int (a + b\csc(e + fx))^m \, dx$$

Verification is Not applicable to the result.

```
[In] Integrate[(a + b*Csc[e + f*x])^m,x]
```

```
[Out] Integrate[(a + b*Csc[e + f*x])^m, x]
```

**fricas [A]** time = 0.57, size = 0, normalized size = 0.00

$$\text{integral}\left(\left(b\csc\left(fx + e\right) + a\right)^m, x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m,x, algorithm="fricas")
```

```
[Out] integral((b*csc(f*x + e) + a)^m, x)
```

**giac [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx + e\right) + a\right)^m \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m,x, algorithm="giac")
```

```
[Out] integrate((b*csc(f*x + e) + a)^m, x)
```

**maple [A]** time = 1.24, size = 0, normalized size = 0.00

$$\int \left(a + b\csc\left(fx + e\right)\right)^m \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+b*csc(f*x+e))^m,x)
```

```
[Out] int((a+b*csc(f*x+e))^m,x)
```

**maxima [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^{m}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

```
[Out] integrate((b*csc(f*x + e) + a)^m, x)
```

**mupad [A]** time = 0.00, size = -1, normalized size = -0.07

$$\int \left(a+\frac{b}{\sin\left(e+f\,x\right)}\right)^{m}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b/sin(e + f*x))^m,x)
```

```
[Out] int((a + b/sin(e + f*x))^m, x)
```

**sympy [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a+b\csc\left(e+fx\right)\right)^{m}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))**m,x)
```

```
[Out] Integral((a + b*csc(e + f*x))**m, x)
```

## 3.58 $\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx$

**Optimal**. Leaf size=22

$$\mathrm{Int}\left(\sin(e+fx)(a+b\csc(e+fx))^m,x\right)$$

```
[Out] Unintegrable((a+b*csc(f*x+e))^m*sin(f*x+e),x)
```

**Rubi** [A] time = 0.03, antiderivative size = 0, normalized size of antiderivative = 0.00, number of steps used = 0, number of rules used = 0, integrand size = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {}

$$\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx$$

Verification is Not applicable to the result.

```
[In] Int[(a + b*Csc[e + f*x])^m*Sin[e + f*x],x]
[Out] Defer[Int][(a + b*Csc[e + f*x])^m*Sin[e + f*x], x]
```

Rubi steps

$$\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx = \int (a+b\csc(e+fx))^m \sin(e+fx)\,dx$$

**Mathematica** [A] time = 6.36, size = 0, normalized size = 0.00

$$\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x],x]
[Out] Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x], x]
```

**fricas** [A] time = 0.50, size = 0, normalized size = 0.00

$$\mathrm{integral}\left(\left(b\csc\left(fx+e\right)+a\right)^m\sin\left(fx+e\right),x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m*sin(f*x+e),x, algorithm="fricas")
[Out] integral((b*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

**giac** [A] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^m \sin\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m*sin(f*x+e),x, algorithm="giac")
[Out] integrate((b*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

**maple** [A] time = 1.07, size = 0, normalized size = 0.00

$$\int \left(a+b\csc\left(fx+e\right)\right)^m \sin\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+b*csc(f*x+e))^m*sin(f*x+e),x)
[Out] int((a+b*csc(f*x+e))^m*sin(f*x+e),x)
```

**maxima [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^{m}\sin\left(fx+e\right)\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m*sin(f*x+e),x, algorithm="maxima")
[Out] integrate((b*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

**mupad [A]** time = 0.00, size = -1, normalized size = -0.05

$$\int \sin\left(e+f\,x\right)\,{\left(a+\frac{b}{\sin\left(e+f\,x\right)}\right)}^m \,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(e + f*x)*(a + b/sin(e + f*x))^m,x)
[Out] int(sin(e + f*x)*(a + b/sin(e + f*x))^m, x)
```

**sympy [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a+b\csc\left(e+fx\right)\right)^{m}\sin\left(e+fx\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))**m*sin(f*x+e),x)
[Out] Integral((a + b*csc(e + f*x))**m*sin(e + f*x), x)
```

## 3.59 $\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx$

**Optimal**. Leaf size=24

$$\operatorname{Int}\left(\sin^2(e+fx)(a+b\csc(e+fx))^m,x\right)$$

```
[Out] Unintegrable((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

**Rubi** [A] time = 0.04, antiderivative size = 0, normalized size of antiderivative = 0.00, number of steps used = 0, number of rules used = 0, integrand size = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {}

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx$$

Verification is Not applicable to the result.

```
[In] Int[(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
[Out] Defer[Int][(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2, x]
```

Rubi steps

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx$$

**Mathematica** [A] time = 6.24, size = 0, normalized size = 0.00

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx$$

Verification is Not applicable to the result.

```
[In] Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
[Out] Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2, x]
```

**fricas** [A] time = 0.51, size = 0, normalized size = 0.00

$$\operatorname{integral}\left(-\left(\cos\left(fx+e\right)^2-1\right)\left(b\csc\left(fx+e\right)+a\right)^m,x\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="fricas")
[Out] integral(-(cos(f*x + e)^2 - 1)*(b*csc(f*x + e) + a)^m, x)
```

**giac** [A] time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^m \sin\left(fx+e\right)^2\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="giac")
[Out] integrate((b*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

**maple** [A] time = 4.45, size = 0, normalized size = 0.00

$$\int \left(a+b\csc\left(fx+e\right)\right)^m \left(\sin^2\left(fx+e\right)\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

```
[Out] int((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

**maxima [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(b\csc\left(fx+e\right)+a\right)^m \sin\left(fx+e\right)^2\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="maxima")
```

```
[Out] integrate((b*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

**mupad [A]** time = 0.00, size = -1, normalized size = -0.04

$$\int \sin\left(e+f\,x\right)^2\,{\left(a+\frac{b}{\sin\left(e+f\,x\right)}\right)}^m\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(sin(e + f*x)^2*(a + b/sin(e + f*x))^m,x)
```

```
[Out] int(sin(e + f*x)^2*(a + b/sin(e + f*x))^m, x)
```

**sympy [A]** time = 0.00, size = 0, normalized size = 0.00

$$\int \left(a+b\csc\left(e+fx\right)\right)^m \sin^2\left(e+fx\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+b*csc(f*x+e))**m*sin(f*x+e)**2,x)
```

```
[Out] Integral((a + b*csc(e + f*x))**m*sin(e + f*x)**2, x)
```

# Chapter 4

# Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

### 4.0.1 Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)


(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*          GradeAntiderivative[result,optimal] returns*)
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)


GradeAntiderivative[result_,optimal_] :=
  If[ExpnType[result]<=ExpnType[optimal],
    If[FreeQ[result,Complex] || Not[FreeQ[optimal,Complex]],
      If[LeafCount[result]<=2*LeafCount[optimal],
        "A",
      "B"],
    "C"],
  If[FreeQ[result,Integrate] && FreeQ[result,Int],
    "C",
  "F"]]


(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
```

```
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]


ElementaryFunctionQ[func_] :=
  MemberQ[{
        Exp,Log,
        Sin,Cos,Tan,Cot,Sec,Csc,
        ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
        Sinh,Cosh,Tanh,Coth,Sech,Csch,
        ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
        Erf, Erfc, Erfi,
        FresnelS, FresnelC,
        ExpIntegralE, ExpIntegralEi, LogIntegral,
        SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
        Gamma, LogGamma, PolyGamma,
        Zeta, PolyLog, ProductLog,
        EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]
```

```
AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

### 4.0.2 Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems

GradeAntiderivative := proc(result,optimal)
local leaf_count_result, leaf_count_optimal,ExpnType_result,ExpnType_optimal,
   debug:=false;

      leaf_count_result:=leafcount(result);
      #do NOT call ExpnType() if leaf size is too large. Recursion problem
      if leaf_count_result > 500000 then
          return "B";
      fi;

      leaf_count_optimal:=leafcount(optimal);

      ExpnType_result:=ExpnType(result);
      ExpnType_optimal:=ExpnType(optimal);

      if debug then
         print("ExpnType_result",ExpnType_result," ExpnType_optimal=",
   ExpnType_optimal);
      fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
#       antiderivative
#   "A" if result can be considered optimal

  #This check below actually is not needed, since I only
  #call this grading only for passed integrals. i.e. I check
  #for "F" before calling this. But no harm of keeping it here.
  #just in case.


  if not type(result,freeof('int')) then
     return "F";
  end if;


  if ExpnType_result<=ExpnType_optimal then
      if debug then
          print("ExpnType_result<=ExpnType_optimal");
      fi;
```

```
      if is_contains_complex(result) then
         if is_contains_complex(optimal) then
            if debug then
               print("both result and optimal complex");
            fi;
            #both result and optimal complex
            if leaf_count_result<=2*leaf_count_optimal then
               return "A";
            else
               return "B";
            end if
         else #result contains complex but optimal is not
            if debug then
               print("result contains complex but optimal is not");
            fi;
            return "C";
         end if
      else # result do not contain complex
           # this assumes optimal do not as well
           if debug then
               print("result do not contain complex, this assumes optimal do
   not as well");
           fi;
           if leaf_count_result<=2*leaf_count_optimal then
              if debug then
                 print("leaf_count_result<=2*leaf_count_optimal");
              fi;
              return "A";
           else
              if debug then
                 print("leaf_count_result>2*leaf_count_optimal");
              fi;
              return "B";
           end if
      end if
  else #ExpnType(result) > ExpnType(optimal)
      if debug then
         print("ExpnType(result) > ExpnType(optimal)");
      fi;
      return "C";
  end if

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
```

```
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:
```

```
AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

### 4.0.3 Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#              Port of original Maple grading function by
#              Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#              added 'exp_polar'
from sympy import *

def leaf_count(expr):
    #sympy do not have leaf count function. This is approximation
    return round(1.7*count_ops(expr))

def is_sqrt(expr):
    if isinstance(expr,Pow):
        if expr.args[1] == Rational(1,2):
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
            asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
            asinh,acosh,atanh,acoth,asech,acsch
        ]

def is_special_function(func):
    return func in [ erf,erfc,erfi,
             fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
             gamma,loggamma,digamma,zeta,polylog,LambertW,
             elliptic_f,elliptic_e,elliptic_pi,exp_polar
         ]

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]
```

```
def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
           return True
        else:
           return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
       return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn))   #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow):   #type(expn,'`^`')
        if isinstance(expn.args[1],Integer):  #type(op(2,expn),'integer')
            return expnType(expn.args[0])   #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational):  #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn
   )))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,
   ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or
   type(expn,'`*`')
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):  #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))  #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)   #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,
   expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType
   ,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9
```

```
#main function
def grade_antiderivative(result,optimal):

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        return "F"

    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    return "A"
                else:
                    return "B"
            else: #result contains complex but optimal is not
                return "C"
        else: # result do not contain complex, this assumes optimal do not as
   well
            if leaf_count_result <= 2*leaf_count_optimal:
                return "A"
            else:
                return "B"
    else:
        return "C"
```

### 4.0.4 SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#              Albert Rich to use with Sagemath. This is used to
#              grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#              'arctan2','floor','abs','log_integral'

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
```

```
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    debug=False
    m = func.name() in ['exp','log','ln',
            'sin','cos','tan','cot','sec','csc',
            'arcsin','arccos','arctan','arccot','arcsec','arccsc',
            'sinh','cosh','tanh','coth','sech','csch',
            'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
        'arctan2','floor','abs'
        ]
    if debug:
        if m:
            print ("func ", func , " is elementary_function")
        else:
            print ("func ", func , " is NOT elementary_function")


    return m

def is_special_function(func):
    debug=False
    if debug: print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
           'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','
    sinh_integral'
           'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
           'polylog','lambert_w','elliptic_f','elliptic_e',
           'elliptic_pi','exp_integral_e','log_integral']

    if debug:
        print ("m=",m)
        if m:
            print ("func ", func ," is special_function")
        else:
            print ("func ", func ," is NOT special_function")


    return m


def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','
    hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric']   #[appellf1] can't find this in
    sagemath

def is_atom(expn):

    debug=False
    if debug: print ("Enter is_atom")


    #thanks to answer at https://ask.sagemath.org/question/49179/what-is-
    sagemath-equivalent-to-atomic-type-in-maple/
    try:
        if expn.parent() is SR:
```

```
            return expn.operator() is None
        if expn.parent() in (ZZ, QQ, AA, QQbar):
            return expn in expn.parent() # Should always return True
        if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens")
    :
            return expn in expn.parent().base_ring() or expn in expn.parent().
    gens()
        return False

    except AttributeError as error:
        return False


def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list:   #isinstance(expn,list):
        return max(map(expnType, expn))   #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if  type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],
    Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.
    args[0]))
    elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
        if type(expn.operands()[1])==Integer:  #isinstance(expn.args[1],Integer
    )
            return expnType(expn.operands()[0])   #expnType(expn.args[0])
        elif type(expn.operands()[1])==Rational:  #isinstance(expn.args[1],
    Rational)
            if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],
    Rational)
                return 1
            else:
                return max(2,expnType(expn.operands()[0]))  #max(2,expnType(
    expn.args[0]))
        else:
            return max(3,expnType(expn.operands()[0]),expnType(expn.operands()
    [1])) #max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1]))
    elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #
    isinstance(expn,Add) or isinstance(expn,Mul)
        m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
        m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.operator()):  #is_elementary_function(expn
    .func)
        return max(3,expnType(expn.operands()[0]))
    elif is_special_function(expn.operator()): #is_special_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(
    expn.args)))
        return max(4,m1)   #max(4,m1)
    elif is_hypergeometric_function(expn.operator()): #
    is_hypergeometric_function(expn.func)
        m1 = max(map(expnType, expn.operands()))       #max(map(expnType, list(
    expn.args)))
        return max(5,m1)   #max(5,m1)
    elif is_appell_function(expn.operator()):
```

```
        m1 = max(map(expnType, expn.operands()))       #max(map(expnType, list(
    expn.args)))
        return max(6,m1)   #max(6,m1)
    elif str(expn).find("Integral") != -1: #this will never happen, since it
                #is checked before calling the grading function that is passed.
                #but kept it here.
        m1 = max(map(expnType, expn.operands()))       #max(map(expnType, list(
    expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):

    if debug: print ("Enter grade_antiderivative for sagemath")

    leaf_count_result  = tree_size(result) #leaf_count(result)
    leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

    if debug: print ("leaf_count_result=", leaf_count_result, "
    leaf_count_optimal=",leaf_count_optimal)


    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",
    expnType_optimal)

    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    return "A"
                else:
                    return "B"
            else: #result contains complex but optimal is not
                return "C"
        else: # result do not contain complex, this assumes optimal do not as
    well
            if leaf_count_result <= 2*leaf_count_optimal:
                return "A"
            else:
                return "B"
    else:
        return "C"
```